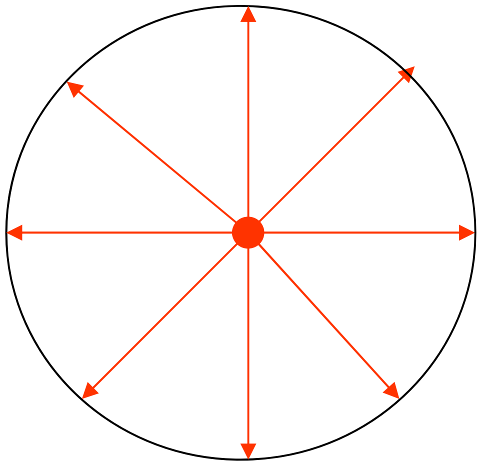
Lect. 7: Diffraction Gratings

Consider isotropic light emission by a point source

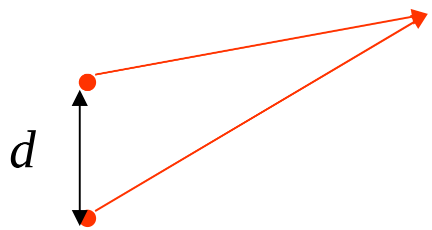


$$E \sim \frac{1}{R} e^{-jkR} \text{ (Spherical wave)}$$

Why $\frac{1}{R}$ dependence?

because $\int |E|^2 R^2 \sin\theta d\theta d\phi$ should be constant.

Two point sources separated by d



$$E_{\text{total}} = ?$$

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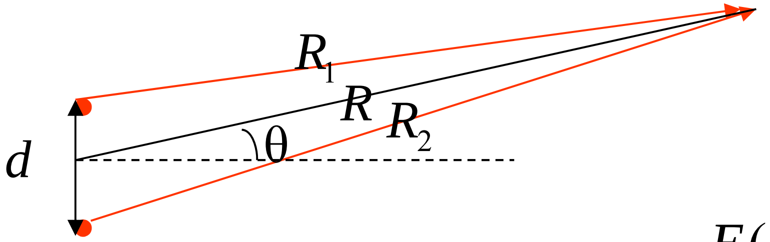

$$E(R,\theta,\phi) = E_1 + E_2$$

$$= \frac{A}{R_1} e^{-jkR_1} + \frac{A}{R_2} e^{-jkR_2}$$

Assume $R >> d$ $\quad R_1 \approx R - \frac{d}{2}\sin\theta \quad R_2 \approx R + \frac{d}{2}\sin\theta$

$$\therefore E(R,\theta) \approx \frac{A}{R} e^{-jk(R-\frac{d}{2}\sin\theta)} + \frac{A}{R} e^{-jk(R+\frac{d}{2}\sin\theta)}$$

$$= \frac{A}{R} e^{-jkR} (e^{jk\frac{d}{2}\sin\theta} + e^{-jk\frac{d}{2}\sin\theta}) = \frac{2A}{R} e^{-jkR} \cos(k\frac{d}{2}\sin\theta)$$

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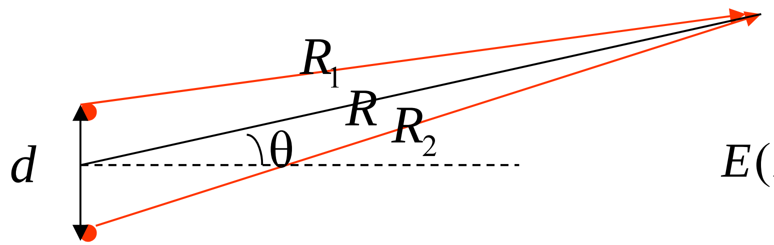

$$E(R,\theta) \approx \frac{2A}{R} e^{-jkR} \cos(k\frac{d}{2}\sin\theta)$$

$$I\text{(Intensity)} : |E|^2 = 4(\frac{A}{R})^2 \cos^2(k\frac{d}{2}\sin\theta)$$

=> There exist max. and min. intensity conditions:

For max., $k\frac{d}{2}\sin\theta = m\pi \Rightarrow I = \left(\frac{2A}{R}\right)^2$

phase difference $=2m\pi$; In Phase

length difference: $d\sin\theta = m\lambda$

➔ Constructive interference

For min., $k\frac{d}{2}\sin\theta = (m+\frac{1}{2})\pi \Rightarrow I = 0$

Phase difference $=(2m+1)\pi$; Out of Phase

length difference: $d\sin\theta = (m+\frac{1}{2})\lambda$

➔ Destructive interference

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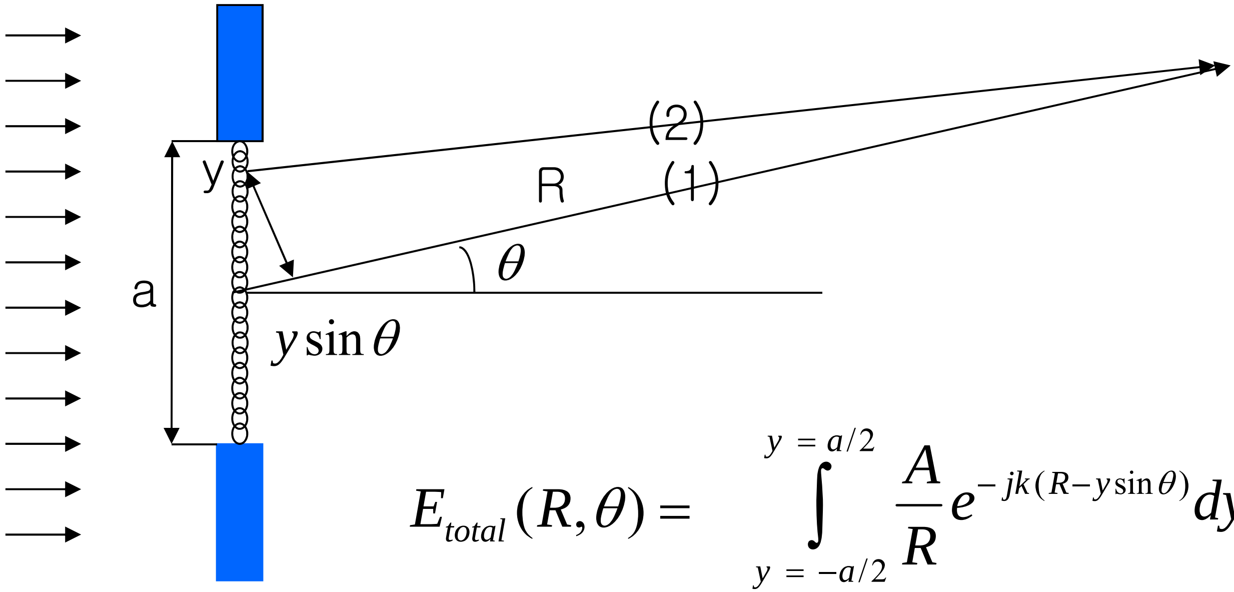

(1) $\dfrac{A}{R}\exp(-jkR)$

(2) $\dfrac{A}{R}\exp\left[-jk(R-y\sin\theta)\right]$

$$E_{total}(R,\theta)=\int_{y=-a/2}^{y=a/2}\frac{A}{R}e^{-jk(R-y\sin\theta)}dy=\frac{A}{R}e^{-jkR}\int_{y=-a/2}^{y=a/2}e^{jky\sin\theta}dy$$

Since interference is determined by ***phase difference*** ,
the constant phase term can be ignored without affecting the final result.

$$E_{total}(R,\theta)=\frac{A}{R}\int_{y=-a/2}^{y=a/2}e^{jky\sin\theta}dy$$

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$$\text{Evaluate } E_{total}(R,\theta) = \frac{A}{R}\int_{-a/2}^{a/2} e^{jky\sin\theta} dy$$

$$\text{Let } y' = jky\sin\theta \;\; => dy' = jk\sin\theta dy$$

$$E_{total}(R,\theta) = \frac{A}{R}\int_{y'=-jk\frac{a}{2}\sin\theta}^{y'=jk\frac{a}{2}\sin\theta} e^{y'} \frac{dy'}{jk\sin\theta} = \frac{A}{R}\frac{1}{jk\sin\theta}\left(e^{jk\frac{a}{2}\sin\theta} - e^{-jk\frac{a}{2}\sin\theta}\right)$$

$$= \frac{A}{R}\frac{2j}{jk\sin\theta}\sin(k\frac{a}{2}\sin\theta)$$

$$= \frac{2A}{R}\frac{\sin(k\frac{a}{2}\sin\theta)}{k\sin\theta}$$

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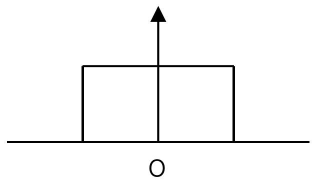
$$E_{total}(R,\theta) = \frac{2A}{R}\frac{\sin(k\frac{a}{2}\sin\theta)}{k\sin\theta}$$

$$\frac{E_{total}(R,\theta)}{E_{total}(R,0)} = ?$$

$$E_{total}(R,0) = \frac{2A}{R}\frac{\cos(k\frac{a}{2}\sin\theta)k\frac{a}{2}\cos\theta\Big|_{\theta=0}}{k\cos\theta\Big|_{\theta=0}} = \frac{2A}{R}\frac{a}{2}$$

$$\therefore \frac{E_{total}(\theta)}{E_{total}(0)} = \frac{\sin(k\frac{a}{2}\sin\theta)}{k\frac{a}{2}\sin\theta} = \frac{\sin\psi}{\psi} \qquad \left(\psi = k\frac{a}{2}\sin\theta\right)$$



↕

sinc function

FT relationship

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$$E_{total}(R,\theta) = \frac{A}{R}\int_{y=-a/2}^{y=a/2} e^{jky\sin\theta}dy \quad \Rightarrow \quad E_{total}(R,k_y) = \frac{1}{R}\int_{y=-\infty}^{y=\infty} A(y)e^{jk_y y}dy$$

$$\left(k_y = k\sin\theta\right)$$

Remember $$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$\mathrm{f}(t) \quad <=> \quad F(\omega)$$

$$E_{total}(k_y) <=> A(y)$$

Far-field diffraction field $E(k_y)$ is F.T. of $A(y)$ within the constant factor!

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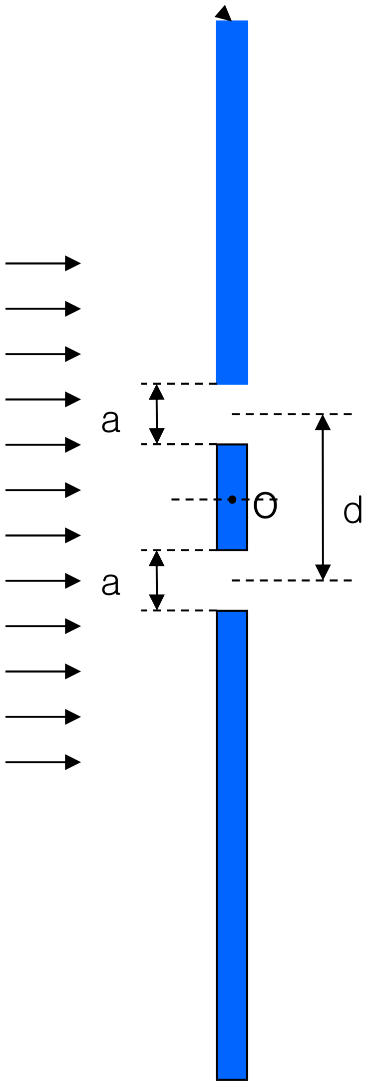

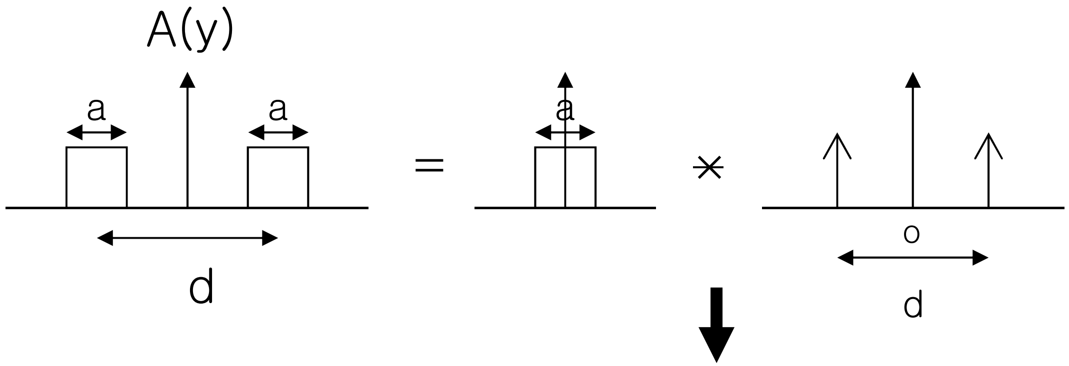

$$\frac{\sin(k_y \frac{a}{2})}{k_y \frac{a}{2}} \quad \text{X} \quad e^{jk_y \frac{d}{2}} + e^{-jk_y \frac{d}{2}} = 2\cos(k_y \frac{d}{2})$$

$$\frac{E(k_y)}{E(0)} = 2\cos(k_y \frac{d}{2}) \frac{\sin(\frac{k_y a}{2})}{\frac{k_y a}{2}}$$

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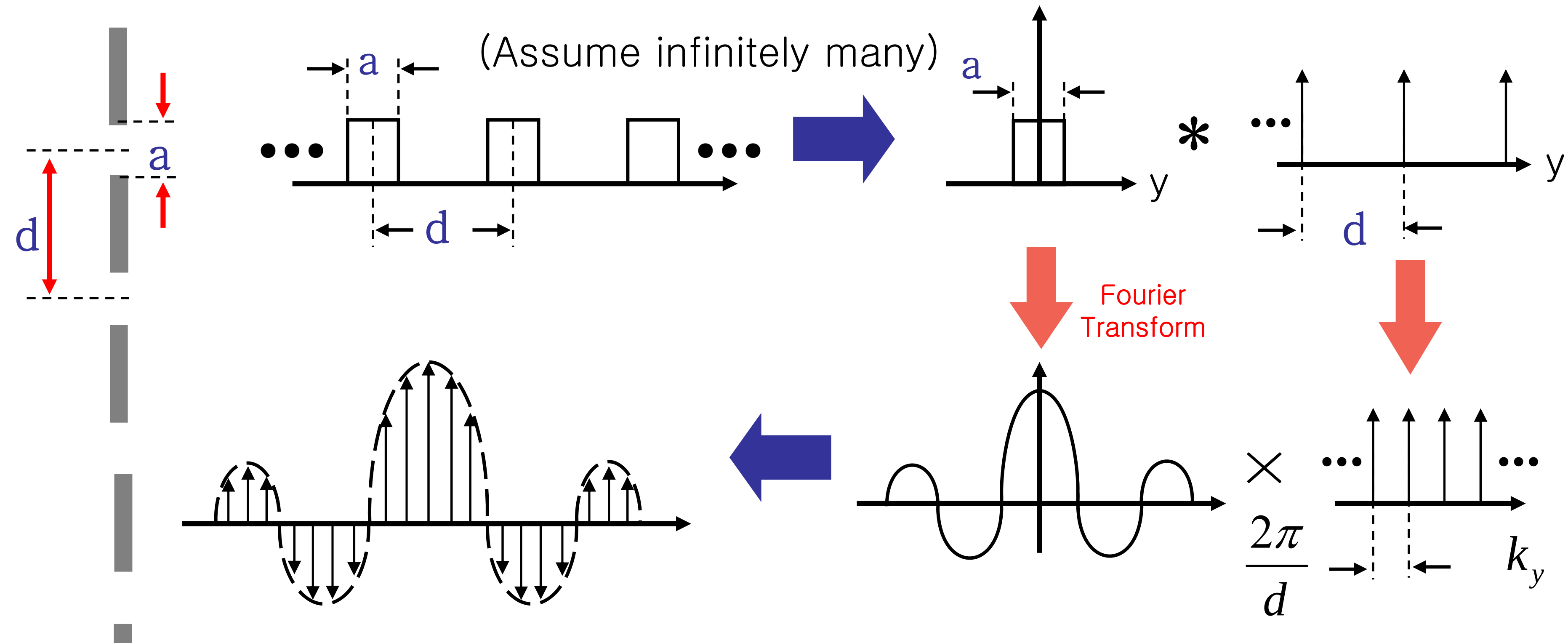

Diffracted light from periodic slits (Grating) => Far-field only at discrete angles

$$\frac{\sin\theta}{\lambda} = m \cdot \frac{1}{d}; \quad d\sin\theta = m\lambda$$

Grating equation, Bragg Condition

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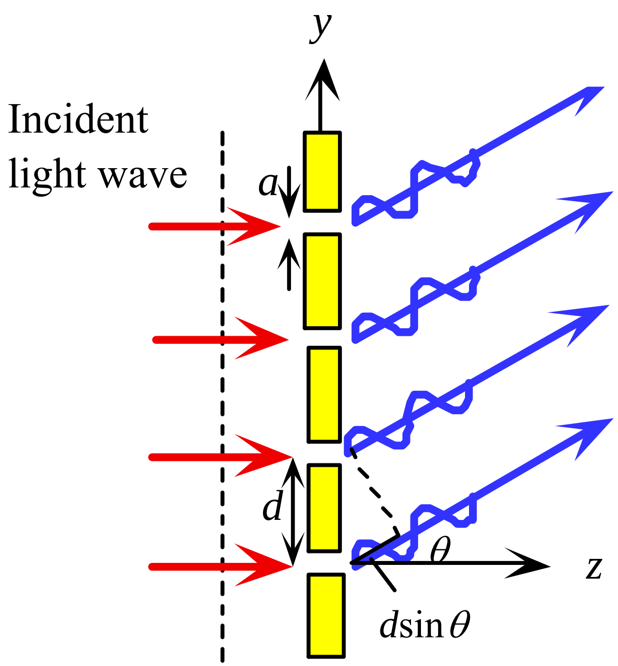
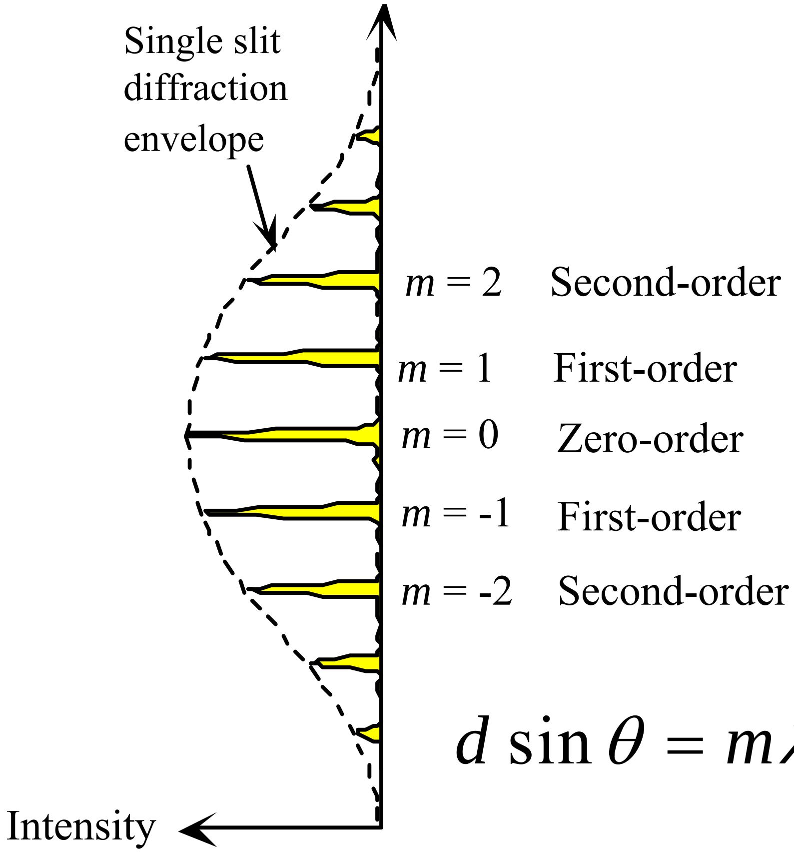


Diffraction grating



$$d \sin\theta = m\lambda$$

Width for each diffracted beam?

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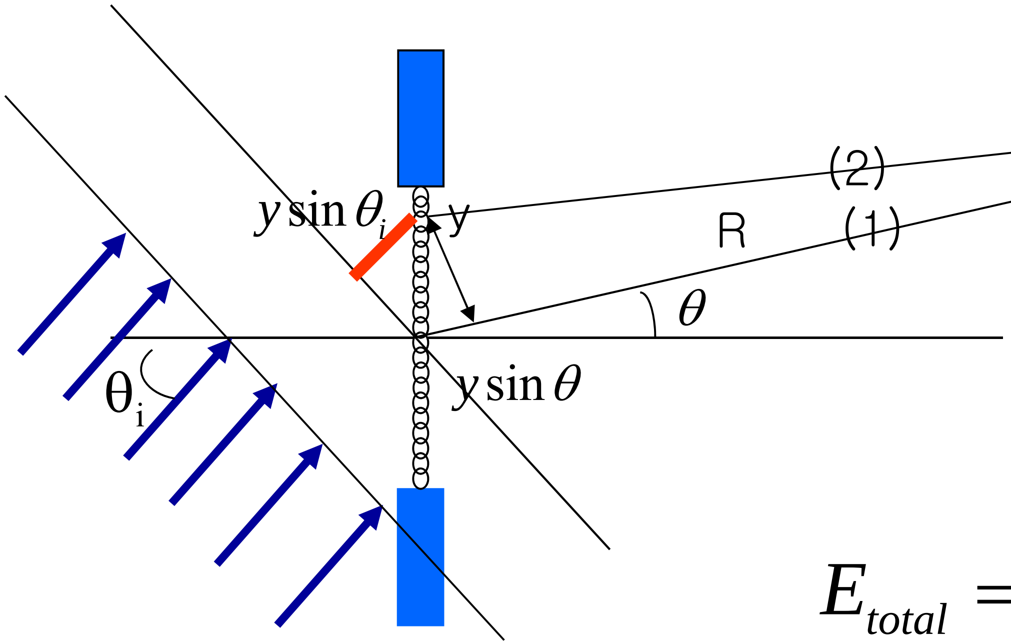
Input with tilted angle



$$\frac{A}{R}e^{jky(\sin\theta-\sin\theta_i)}$$

$$E_{total}=\int_{-\infty}^{\infty}\frac{A}{R}e^{jky(\sin\theta-\sin\theta_i)}dy$$

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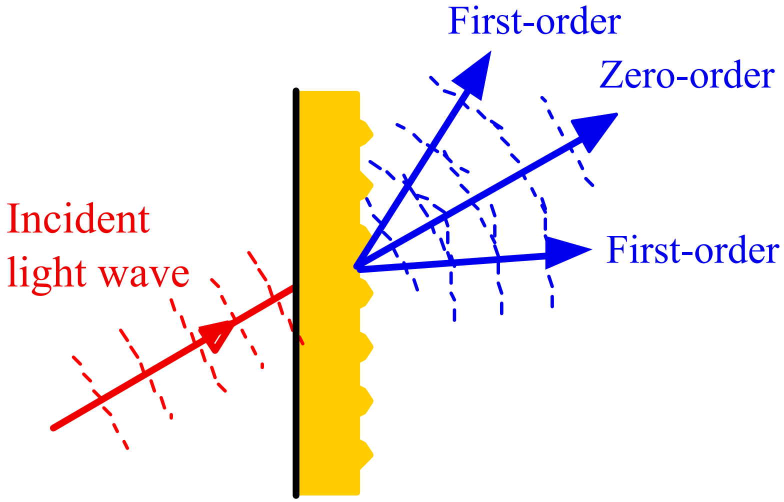
Tilted incidence on grating



$$d \sin\theta = m\lambda$$

$$\Rightarrow \quad d(\sin\theta - \sin\theta_i) = m \cdot \lambda$$

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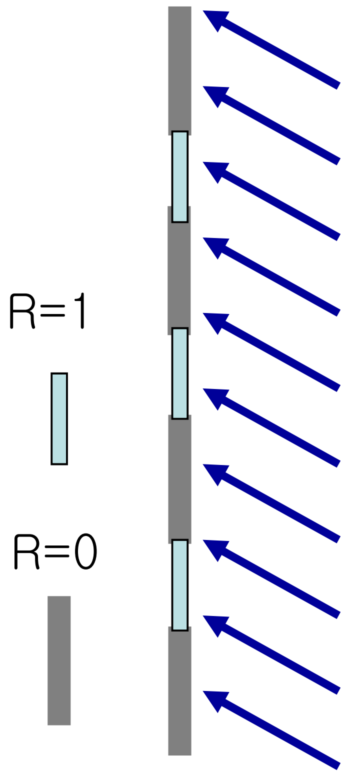
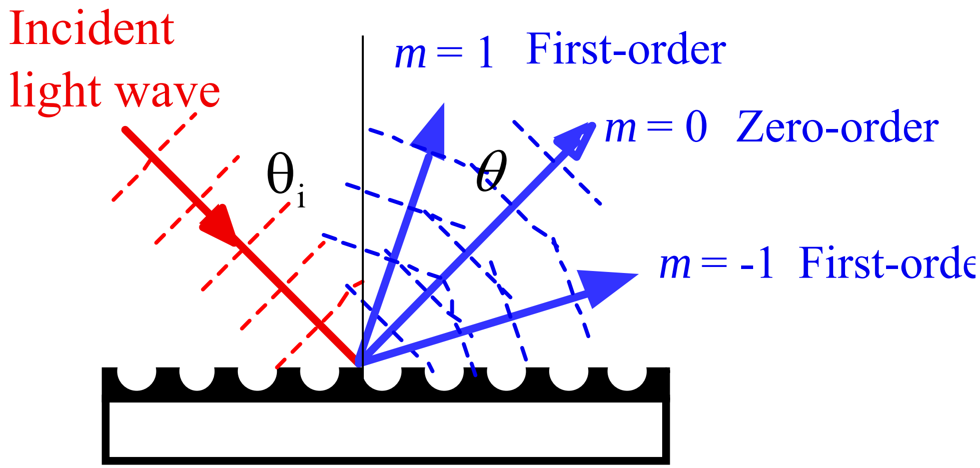
Reflection-type grating



Same diffraction equation applies

$$d\left(\sin\theta - \sin\theta_i\right) = m \cdot \lambda$$



(b) Reflection grating

Grating can be realized as long as reflection surface is periodic

Gratings are widely used as λ demultiplexer

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Determine grating conditions for following two cases.