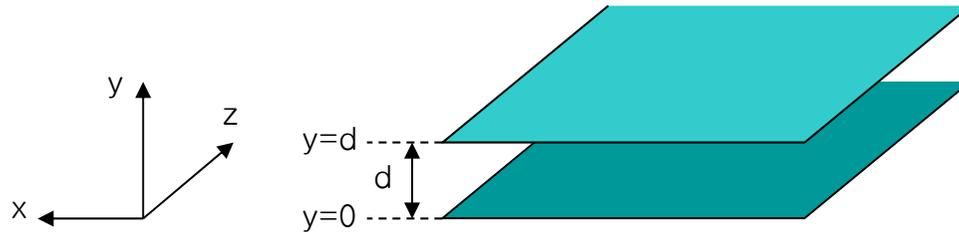


# Lect. 23 TM waves (Cheng 10-3.1)

$E_z \neq 0, H_z = 0$  : TM (Transverse Magnetic) waves



$$\nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2}{\partial t^2} \bar{E} = 0$$

$$\bar{E}(x, y, z, t) = \bar{E}(x, y) e^{-j\gamma z} e^{j\omega t}$$

$$\gamma = \alpha + j\beta$$

$$\nabla_{x,y}^2 \bar{E}(x, y) + h^2 \bar{E}(x, y) = 0$$

$$h^2 = \gamma^2 + \omega^2 \mu\epsilon$$

$$\nabla_{x,y}^2 E_z(x, y) + h^2 E_z(x, y) = 0$$

$$\frac{d^2}{dy^2} E_z(y) + h^2 E_z(y) = 0$$

Solution :  $E_z(y) = A \sin(hy) + B \cos(hy)$

Boundary conditions:  $E_z(y) = 0$  at  $y=0$  (1) and  $y=d$  (2)

$$\nabla_{x,y}^2 E_z(x, y) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E_z(x, y)$$

(1)  $B = 0$

(2)  $A \sin(hd) = 0 \quad hd = n\pi \text{ or } h = \frac{n\pi}{d} \quad \therefore E_z(y) = A_n \sin\left(\frac{n\pi}{b} y\right)$

# Lect. 23 TM waves

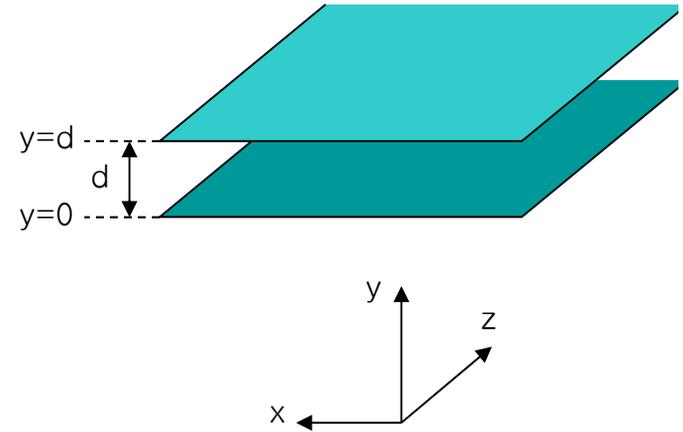
Remember

$$H_x = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z}{\partial x} - j\omega\epsilon \frac{\partial E_z}{\partial y} \right)$$

$$H_y = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z}{\partial y} + j\omega\epsilon \frac{\partial E_z}{\partial x} \right)$$

$$E_x = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z}{\partial x} + j\omega\mu \frac{\partial H_z}{\partial y} \right)$$

$$E_y = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z}{\partial y} - j\omega\mu \frac{\partial H_z}{\partial x} \right)$$



$$E_z(y) = A_n \sin(hy), \quad h = \frac{n\pi}{b}$$

$$E_x(y) = 0$$

$$E_y(y) = -\frac{\gamma}{h} A_n \cos(hy) = -\frac{\gamma}{n\pi/b} A_n \cos\left(\frac{n\pi}{b} y\right)$$

$$H_z(y) = 0$$

$$H_x(y) = -\frac{1}{h} (-j\omega\epsilon) A_n \cos(hy) = \frac{j\omega\epsilon}{n\pi/b} A_n \cos\left(\frac{n\pi}{b} y\right)$$

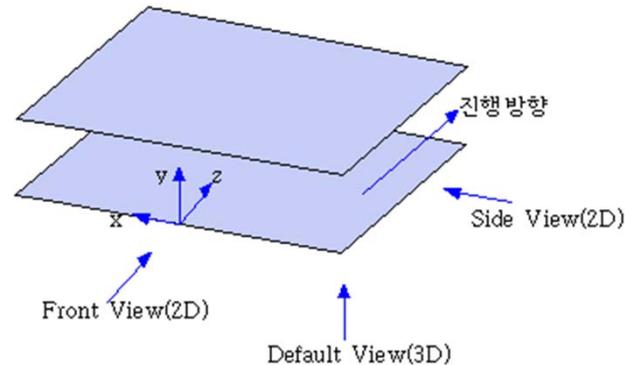
$$H_y(y) = 0$$

Different types of solutions depending on n  $\Rightarrow$  Mode:  $TM_n$

For  $TM_0 \Rightarrow h = n\pi/b = 0$  Only  $E_y$  and  $H_x$  components exist  $\rightarrow$  TEM wave

# Lect. 23 TM waves

Wave propagation animation for TM<sub>1</sub> and TM<sub>2</sub>



$$E_x(y) = 0$$

$$E_y(y) = -\frac{\gamma}{n\pi/b} A_n \cos\left(\frac{n\pi}{b} y\right)$$

$$E_z(y) = A_n \sin\left(\frac{n\pi}{b} y\right)$$

$$H_x(y) = \frac{j\omega\epsilon}{n\pi/b} A_n \cos\left(\frac{n\pi}{b} y\right)$$

$$H_y(y) = 0$$

$$H_z(y) = 0$$

For following animations,  $\bar{E}(x, y, z, t) = \text{Re} \left[ \bar{E}(x, y) e^{-j\beta z} e^{j\omega t} \right]$

$\bar{H}(x, y, z, t) = \text{Re} \left[ \bar{H}(x, y) e^{-j\beta z} e^{j\omega t} \right]$

---

---

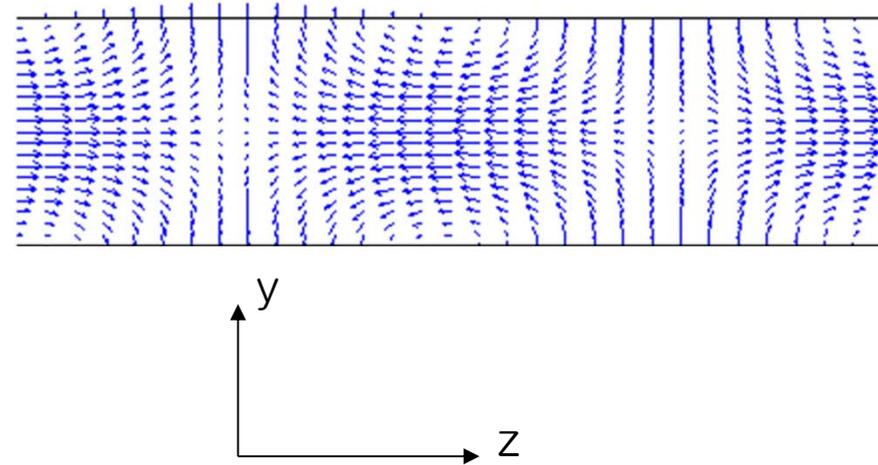
Side View

TM<sub>1</sub>

$$E_x(y, z, t) = 0$$

$$E_y(y, z, t) = \frac{\beta b}{\pi} A_1 \cos\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z)$$

$$E_z(y, z, t) = A_1 \sin\left(\frac{\pi}{b} y\right) \cos(\omega t - \beta z)$$

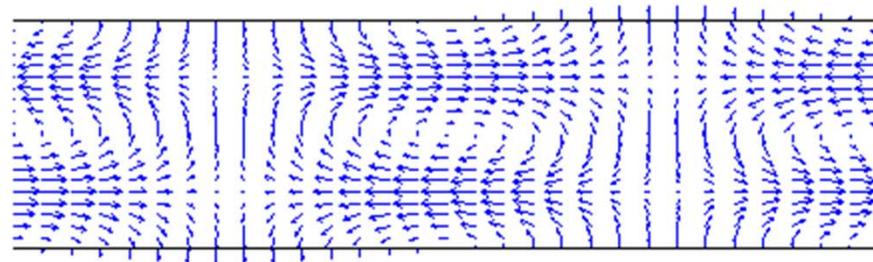


TM<sub>2</sub>

$$E_x(y, z, t) = 0$$

$$E_y(y, z, t) = \frac{\beta b}{2\pi} A_2 \cos\left(\frac{2\pi}{b} y\right) \sin(\omega t - \beta z)$$

$$E_z(y, z, t) = A_2 \sin\left(\frac{2\pi}{b} y\right) \cos(\omega t - \beta z)$$



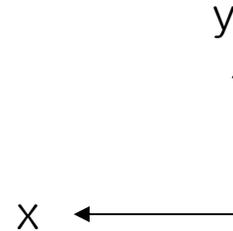
---

---

Front View at fixed z

TM<sub>1</sub>

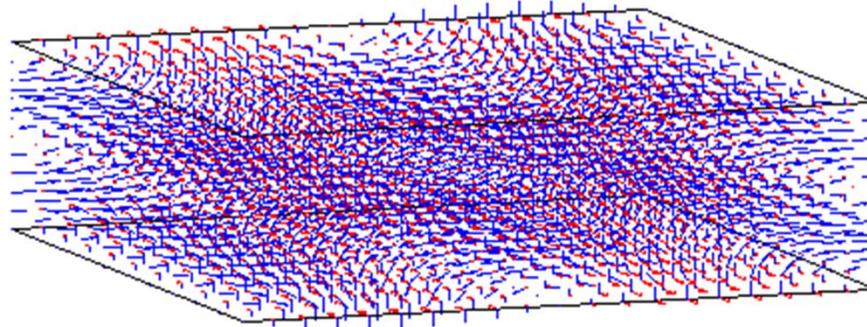
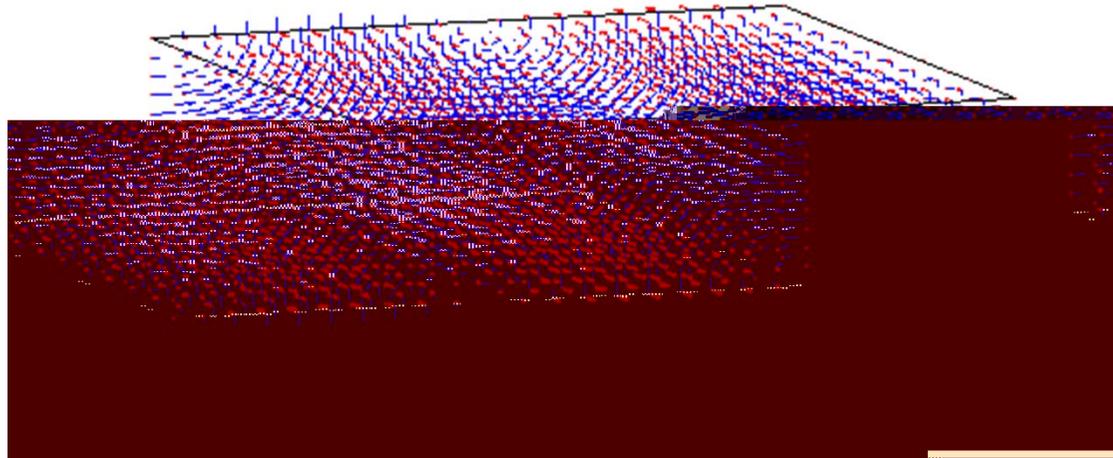
$$H_x(y, z, t) = -A_1 \frac{\omega \epsilon b}{\pi} \cos\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z)$$



TM<sub>2</sub>

$$H_x(y, z, t) = -A_2 \frac{\omega \epsilon b}{2\pi} \cos\left(\frac{2\pi}{b} y\right) \sin(\omega t - \beta z)$$



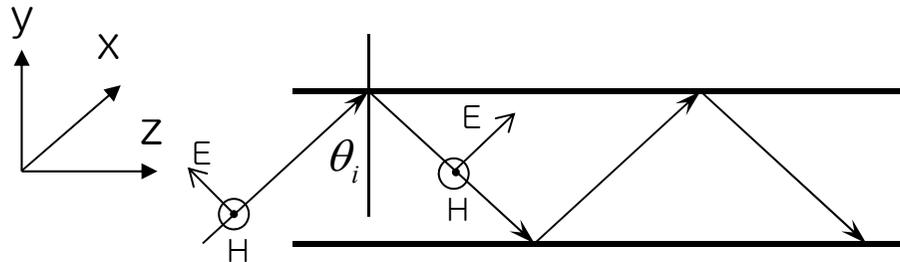


# Lect. 23 TM waves

For  $TM_n$  with  $n > 0$ , ( $h = \frac{n\pi}{d}$ )      Use plane-wave interpretation

$$E_z(y, z) = A \sin(hy) e^{-j\beta z} = \frac{A}{2j} (e^{jhy} - e^{-jhy}) e^{-j\beta z} = \frac{A}{2j} [e^{-j(\beta z - hy)} - e^{-j(\beta z + hy)}]$$

Two plane waves propagating in (+z, -y) and (+z, +y) directions



=> Oblique incident at conductor  
(parallel polarization)  
Two interfaces: top and bottom

$$h = k \cos \theta_i = \frac{n\pi}{b}$$

$$\beta = k \sin \theta_i = \sqrt{k^2 - h^2} = \sqrt{\omega^2 \mu \epsilon - \frac{n^2 \pi^2}{b^2}}$$

# Lect. 23 TM waves

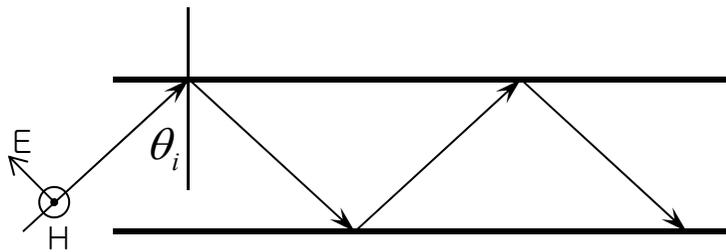
Why are there modes?

Or, why  $h$  cannot be any number but has to satisfy  $h = n\pi/b$ ?

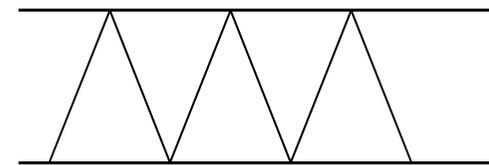
– In  $y$ -direction, wave is going through a periodic motion

→ it should be same after one period:

$$\exp[-j(2hb)] = 1 \quad \rightarrow \quad h = n\pi/b, \quad \text{Discrete } h \text{ (mode)} \quad \Rightarrow \quad \text{discrete } \theta_i$$



$$\begin{cases} \beta = k \sin \theta_i \\ h = k \cos \theta_i \end{cases}$$



Higher mode

# Lect. 23 TM waves

- What happens if  $\omega$  becomes smaller for a given  $TM_n$  mode?

$$\omega \downarrow \implies k \downarrow (\because k = \omega\sqrt{\mu\epsilon}) \implies \theta_i \downarrow \text{ (so that } h \text{ remains constant) until } \theta_i = 0$$

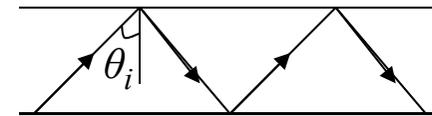
$(h = k, \beta = 0)$

- There exists minimum  $\omega$  for propagation for a given mode

→ cut-off frequency

$$\omega_{\min} \sqrt{\mu\epsilon} = k_{\min} = h = \frac{n\pi}{b}$$

$$f_{\min} = \frac{n}{2b\sqrt{\mu\epsilon}} \implies f_c$$



$$h = k \cos \theta_i$$

$$\beta = k \sin \theta_i$$

What happens if  $f < f_c$ ?

$$\text{From } \beta = \sqrt{\omega^2 \mu\epsilon - \frac{n^2 \pi^2}{b^2}}, \beta = j\alpha \text{ if } \omega < \omega_c.$$

$\exp(-\beta z) = \exp(-\alpha z)$ ; no propagation!

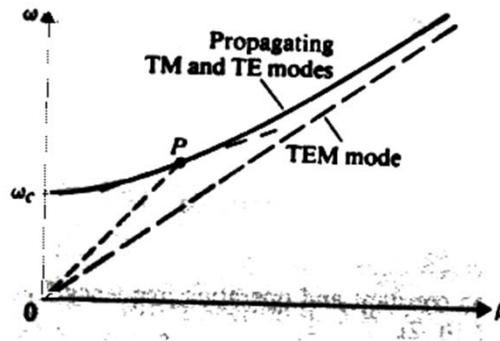
# Lect. 23 TM waves

Dispersion Relationship

$$\beta = \sqrt{k^2 - h^2} = k \sqrt{1 - \frac{h^2}{k^2}}$$

$$= \omega \sqrt{\mu \epsilon} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\frac{h}{k} = \frac{n\pi}{b} \frac{1}{2\pi f \sqrt{\mu \epsilon}} = \frac{n}{2b\sqrt{\mu \epsilon}} \cdot \frac{1}{f} = \frac{f_c}{f}$$



$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \epsilon}} \frac{1}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$v_g = \frac{1}{\frac{d\beta}{d\omega}} = \frac{1}{\sqrt{\mu \epsilon}} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$