Lect. 6: Wave Equations and Plane-Wave Solutions

(Cheng 7.6, 7.7)

Maxwell's Equations

$$\nabla \cdot \overline{D} = \rho$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

$$\nabla \cdot \overline{B} = 0$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$$

$$\overline{D} = \varepsilon \overline{E}$$

$$\overline{B} = \mu \overline{H}$$

✓ Simplification

1) source free medium ➔ $\rho = 0, \ \overline{J} = 0$

2) uniform medium ➔ $\varepsilon, \mu \neq f(x, y, z)$

$$\nabla \times \left(\nabla \times \overline{E}\right) = -\nabla \times \left(\frac{\partial \overline{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \overline{B})$$

$$= -\mu \frac{\partial}{\partial t}(\nabla \times \overline{H}) = -\mu \frac{\partial^2 \overline{D}}{\partial t^2}$$

$$= -\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}$$

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$$\nabla \times \left(\nabla \times \overline{E} \right) = -\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}$$

$$\nabla \times \left(\nabla \times \overline{E} \right) = \nabla \left(\nabla \cdot \overline{E} \right) - \nabla^2 \overline{E}$$

$$\nabla^2 \overline{E} = \overline{x}\ \nabla^2 E_x + \overline{y}\ \nabla^2 E_y + \overline{z}\ \nabla^2 E_z$$ (Vector Laplacian)

$$\nabla^2 E_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}$$ (Laplacian)

From $\nabla \cdot \overline{D} = \rho$, $\quad \nabla \cdot \overline{E} = 0$

$$\Rightarrow \quad \nabla^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}$$ EM Wave Equation!

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$$\nabla^2 \overline{E}(x,y,z,t) = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}(x,y,z,t)$$

-An example of wave equation solutions: $\overline{E} = \overline{x} E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon}\, z)\right]$

- Verification:

- How does the solution look?

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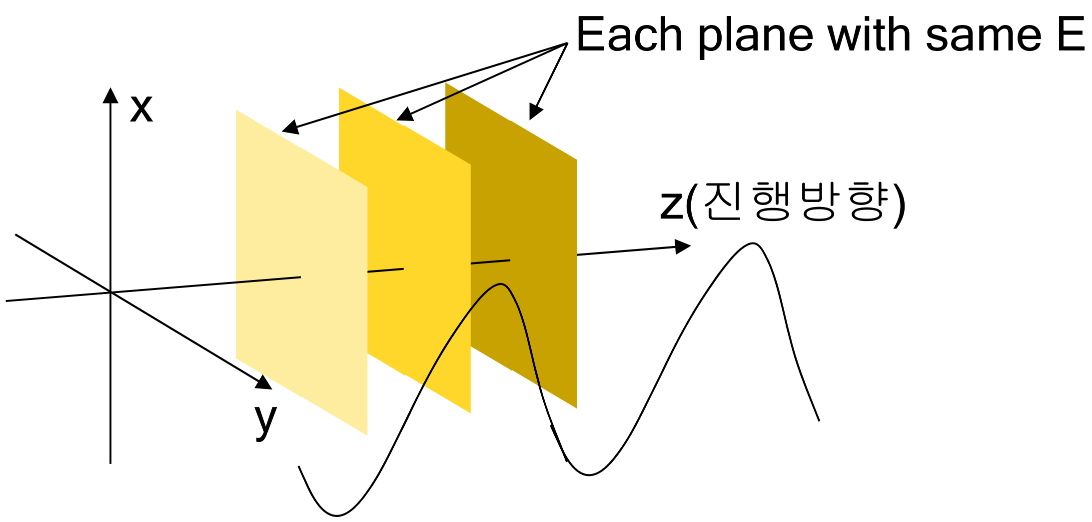
$$\overline{E} = \overline{x}E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon}z)\right]$$



E-field on a x-y plane is uniform

➔ Plane waves

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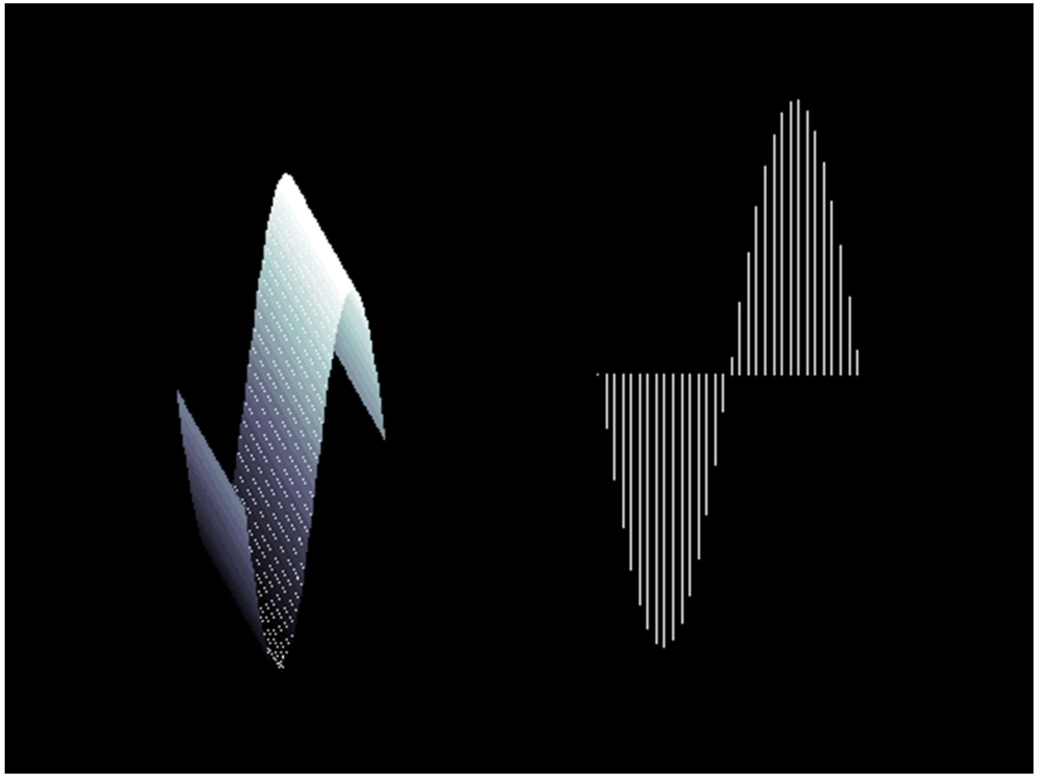
$$\overline{E} = \overline{x}E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon}z)\right]$$



Animations available in can course homepage
(Visualization of EM waves)

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Solve the wave equation more mathematically: $\nabla^2 \overline{E} = \mu\varepsilon \dfrac{\partial^2 \overline{E}}{\partial t^2}$

Assume time dependence of $e^{j\omega t}$ (Time harmonic solutions)

$\overline{E}(x, y, z, t) = \overline{E}(x, y, z) \cdot e^{j\omega t}$ (Separation of variables)

$$\nabla^2 \overline{E}(x, y, z) \cdot e^{j\omega t} + \mu\varepsilon\omega^2 \overline{E}(x, y, z) \cdot e^{j\omega t} = 0$$

$$\nabla^2 \overline{E}(x, y, z) + \mu\varepsilon\omega^2 \overline{E}(x, y, z) = 0$$

$$\text{Let } \omega\sqrt{\mu\varepsilon} = k \quad \nabla^2 \overline{E}(x, y, z) + k^2 \overline{E}(x, y, z) = 0$$

$$k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda} \qquad k : wave\ number \left[m^{-1} \right]$$

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$$\nabla^2 \overline{E}(x,y,z) + k^2 \overline{E}(x,y,z) = 0$$

Assume $\overline{E}(x,y,z) = \overline{x}E\left(z\right)$ for simplicity

$$\boxed{\frac{d^2}{dz^2}E(z) + k^2 E(z) = 0}$$

→ **simple differential equation!**

Solutions?

$$E(z) = E_0^+ e^{-jkz} + E_0^- e^{jkz}$$

$$E(z,t) = (E_0^+ e^{-jkz} + E_0^- e^{jkz})e^{j\omega t} = E_0^+ e^{j(\omega t - kz)} + E_0^- e^{j(\omega t + kz)}$$

How does it look?

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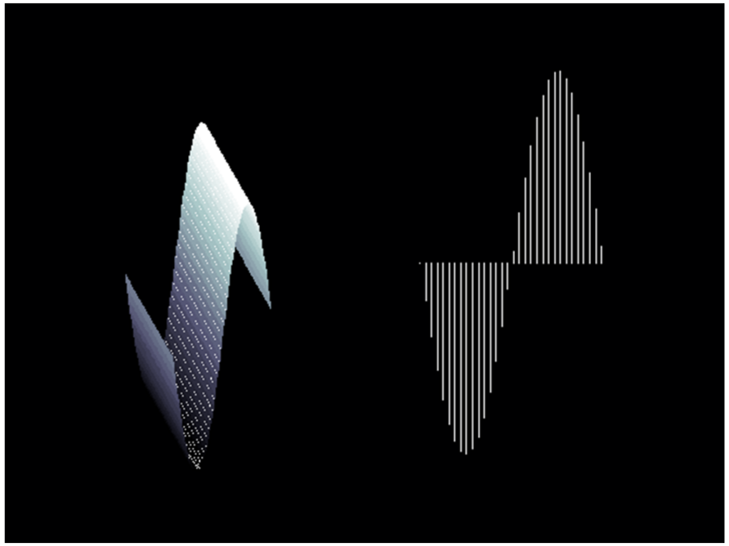
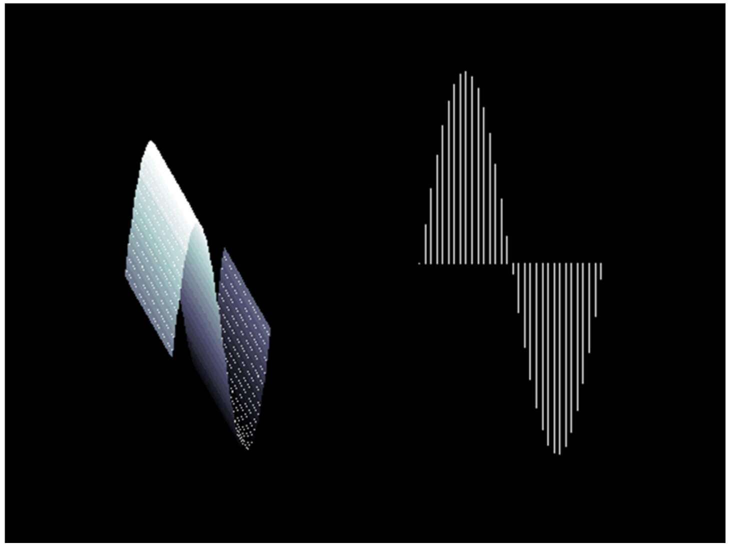
$$\mathrm{Re}\left\{E_0^+ e^{j(\omega t - kz)} + E_0^- e^{j(\omega t + kz)}\right\} = E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz)$$

How does cos(ωt-kz) change as t increases? ➔ Propagation in + z direcition

How does cos(ωt+kz) change as t increases? ➔ Propagation in - zdirecition





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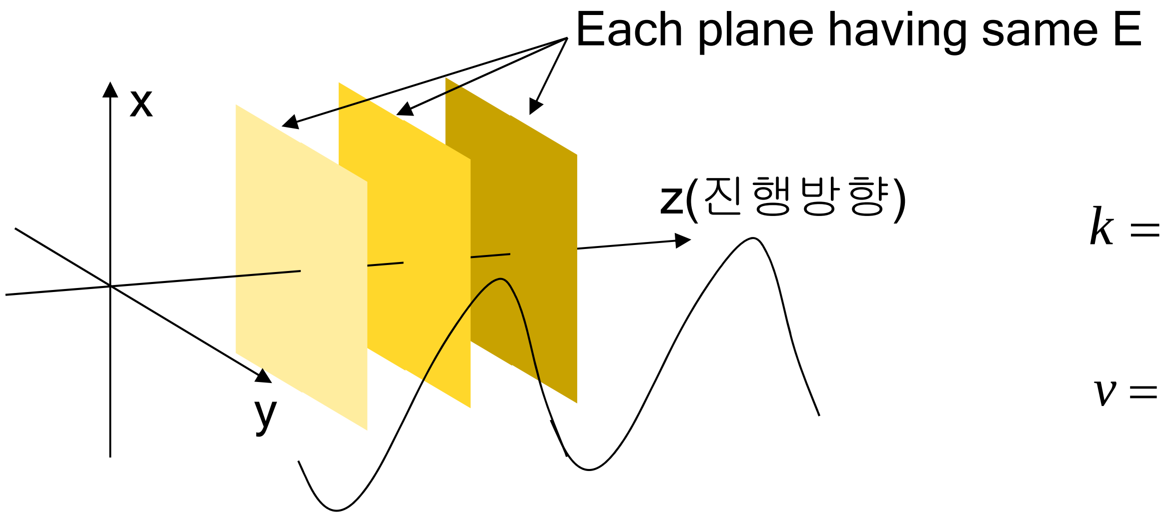
How fast does the wave move? $E_0^+ e^{j(\omega t - kz)}$



$$k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c}$$

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

(Phase velocity)

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How about H-field?

For $\overline{E} = \overline{x}E_o e^{-jkz} e^{j\omega t}$ ➔ Corresponding H-field?

$$\text{From } \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times E = -\mu \frac{\partial \overline{H}}{\partial t} \quad \nabla \times E = -(j\omega)\mu\overline{H}$$

$$\nabla \times E = \begin{vmatrix} \overline{x} & \overline{y} & \overline{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0 e^{-jkz} e^{j\omega t} & 0 & 0 \end{vmatrix} = \overline{y}(-jk)E_0 e^{-jkz} e^{j\omega t}$$

$$\therefore \overline{H} = \overline{y}\frac{k}{\omega\mu}E_0 e^{-jkz} e^{j\omega t} = \overline{y}\sqrt{\frac{\varepsilon}{\mu}}E_0 e^{-jkz} e^{j\omega t}$$

H-field has the same wave characteristics (frequency, wave number, propagation direction) but different vector direction and magnitude

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$$\overline{E} = \overline{x}E_o e^{-jkz} e^{j\omega t} \qquad \overline{H} = \overline{y}\sqrt{\frac{\varepsilon}{\mu}} E_0 e^{-jkz} e^{j\omega t}$$

$$\frac{\left|\overline{E}\right|}{\left|\overline{H}\right|} = \sqrt{\frac{\mu}{\varepsilon}} \equiv \eta$$ (Impedance, about 337Ω in vacuum)

$$\overline{E} = \overline{x}E_o e^{-jkz}$$

$$\overline{H} = \overline{y}H_o e^{-jkz}$$

$$\overline{P} = \overline{E} \times \overline{H}$$ Poynting vector: Propagation of power density

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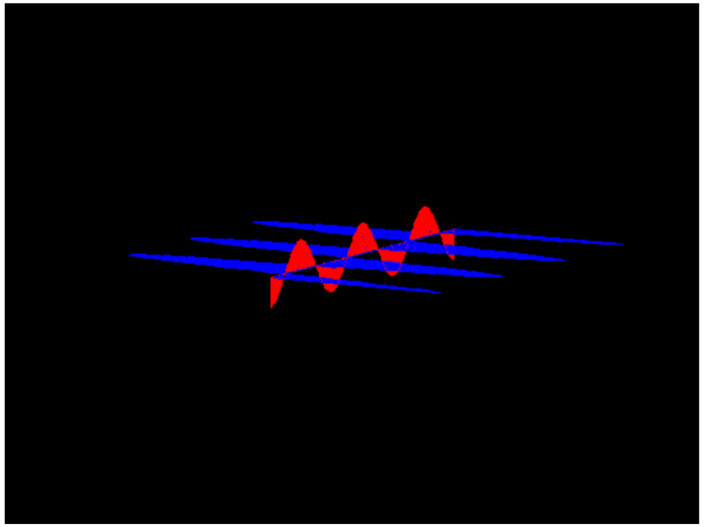
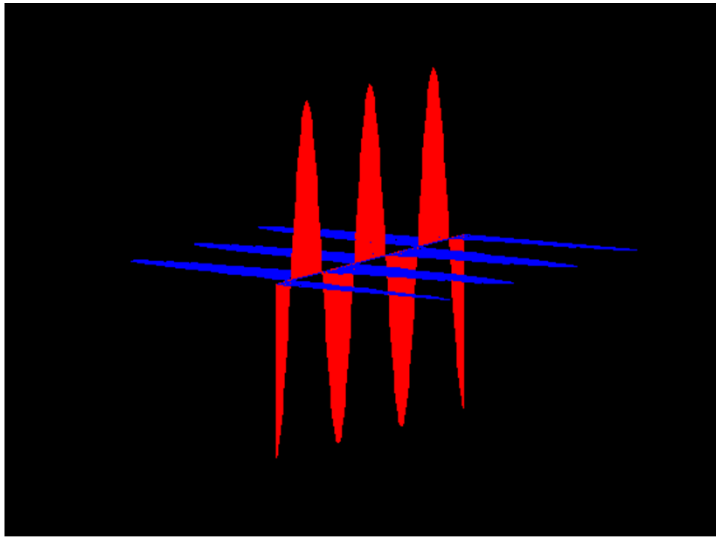
Which has large η ?

(E-field: Blue)





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$$\overline{E} = \overline{x} E_0 e^{-jkz} e^{j\omega t}$$

✓ How do we express waves propagating in other directions?

- In y-direction?

- In x-direction?

  Any problem?

- In y-z direction?

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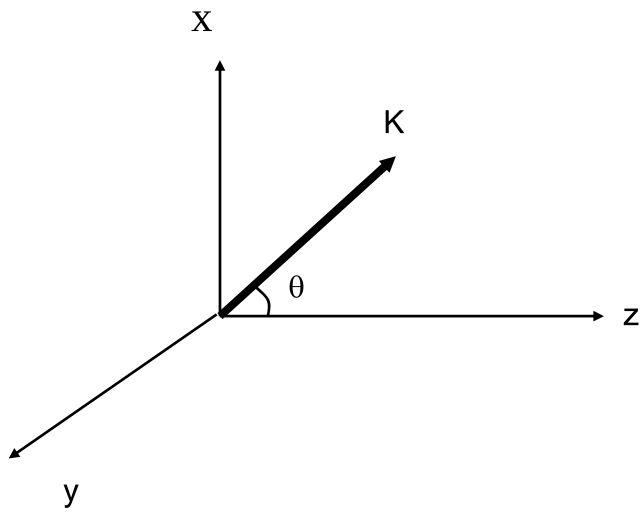
✓ Wave propagating in any direction: Use vector k



$$e^{-jk_x x}\cdot e^{-jk_y y}\cdot e^{-jk_z z} = e^{-j(k_x x + k_y y + k_z z)}$$

$$= e^{-j\bar{k}\cdot\bar{R}}$$

$$\text{With } \bar{R} = \bar{x}x + \bar{y}y + \bar{z}z,$$

$$\bar{k} = \bar{x}k_x + \bar{y}k_y + \bar{z}k_z$$

$\bar{k}$: wave vector

$\angle\bar{k}$: Direction of propagation

$\left|\bar{k}\right|$: $\dfrac{2\pi}{\lambda}$