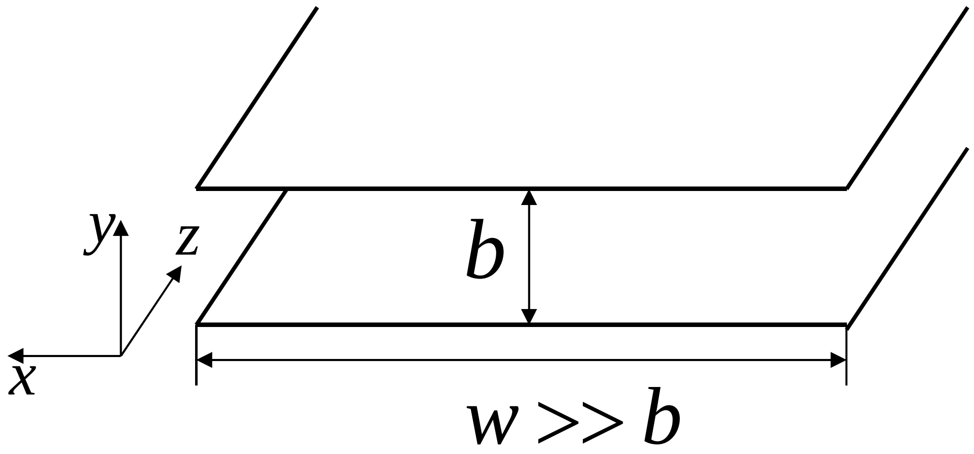
# Lect. 11: Metallic Waveguides

Can we send EM waves without worrying about diffraction?   Waveguides

Consider Metallic Waveguide




Solve the E&M wave equation $\nabla^2 \overline{E} - \mu\varepsilon \frac{\partial^2}{\partial t^2} \overline{E} = 0$

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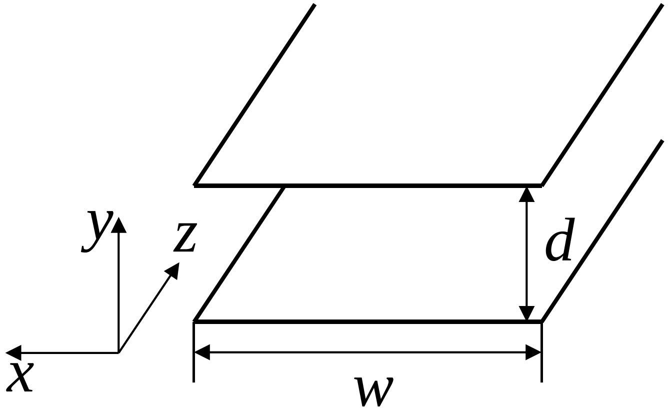


$$\nabla^2 \bar{E} - \mu\varepsilon \frac{\partial^2}{\partial t^2}\bar{E} = 0$$

Assume $\bar{E}(x,y,z,t) = \bar{E}(y)\cdot e^{-j\beta z}\cdot e^{j\omega t}$

$$\nabla^2 \bar{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\bar{E} = \left(\frac{d^2}{dy^2} - \beta^2\right)\bar{E}(y)$$

$$\mu\varepsilon \frac{\partial^2}{\partial t^2}\bar{E} = -\omega^2\mu\varepsilon\bar{E} = -k^2\bar{E} \quad (k^2 = \omega^2\mu\varepsilon)$$

$$\frac{d^2}{dy^2}\bar{E}(y) + (k^2 - \beta^2)\bar{E}(y) = 0$$

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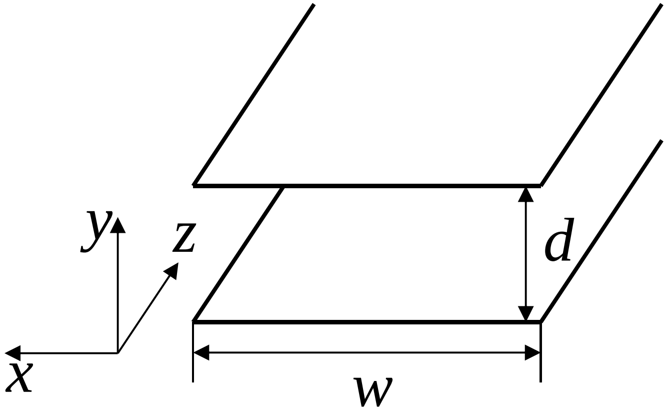


$$\bar{E}(x,y,z,t) = \bar{E}(y)\cdot e^{-j\beta z}\cdot e^{j\omega t}$$

$$\frac{d^2}{dy^2}\bar{E}(y) + (k^2-\beta^2)\bar{E}(y) = 0$$

$$(k^2 = \omega^2\mu\varepsilon)$$

Simplest solution: $\bar{E}(y) = \bar{y}E_0$ with $\beta = k$

$\bar{E}(x,y,z,t) = \bar{y}E_0 e^{-jkz}e^{j\omega t}$ (Plane wave between top and bottom plates)

Boundary conditions?

Corresponding magnetic field? $\bar{H}(x,y,z,t) = -\bar{x}\dfrac{E_0}{\eta}e^{-jkz}e^{j\omega t}$

➔ TEM Solution Transmission Line with *V(x,y,z,t)* and *I (x,y,z,t)*

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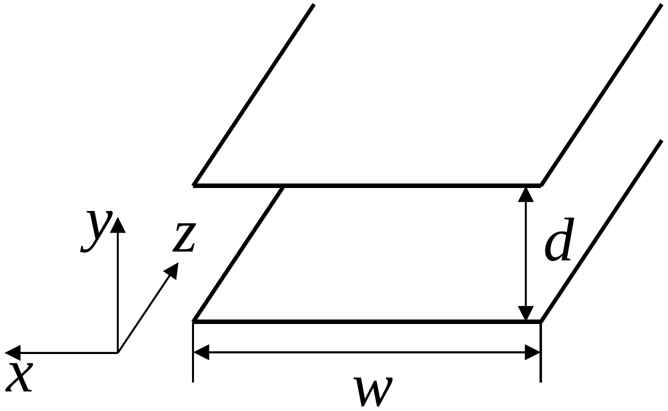


$$\bar{E}(x,y,z,t) = \bar{E}(y) \cdot e^{-j\beta z} \cdot e^{j\omega t}$$

$$\frac{d^2}{dy^2}\bar{E}(y) + (k^2 - \beta^2)\bar{E}(y) = 0$$

Another type of solution: $\bar{E}(y) = \bar{x}E_0 \sin(k_y y)$

$$\bar{E}(x,y,z,t) = \bar{x}E_0 \sin(k_y y) e^{-j\beta z} e^{j\omega t} \quad (\beta^2 = k^2 - k_y{}^2)$$

Boundary conditions? $k_y = \dfrac{m\pi}{d}$ Quantization of $k_y$ and $\beta$

➔ TE Solution

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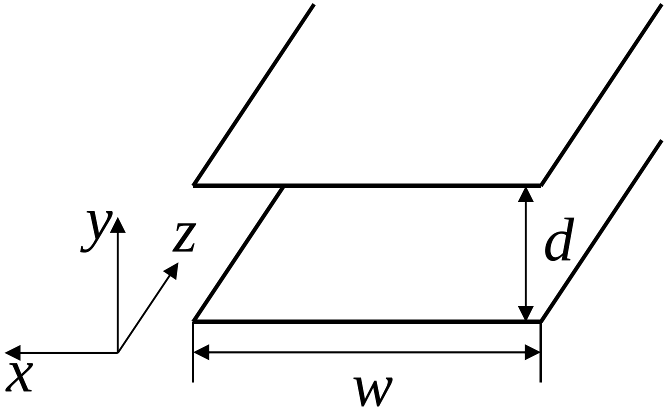


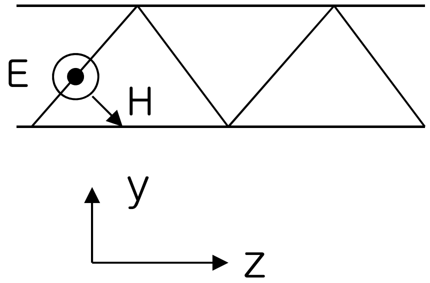

TE Solution

$$\bar{E}(y,z) = \bar{x}E_0 \sin(k_y y)e^{-j\beta z}$$

$$k_y = \frac{m\pi}{d} \qquad \beta^2 = k^2 - \left(\frac{m\pi}{d}\right)^2$$

Interpretation $\sin(k_y y)e^{-j\beta z} \sim (e^{jk_y y} - e^{-jk_y y})e^{-j\beta z}$

➔ two plane waves propagating in different directions

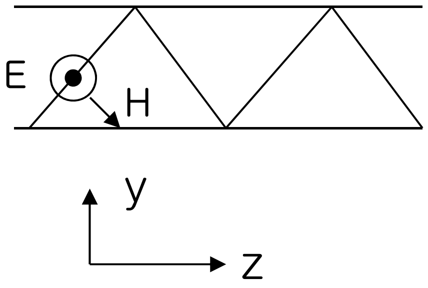



In y-direction, the phase should be the same after one round-trip

$$\exp\left(-j2k_y d\right) = 1 \quad \Rightarrow \quad k_y = \frac{2m\pi}{2d} = \frac{m\pi}{d}$$

(perpendicular polarization)

# Lect. 11: Metallic




TE Solution

$$\bar{E}(y,z) = \bar{x}E_0 \sin(k_y y)e^{-j\beta z}$$

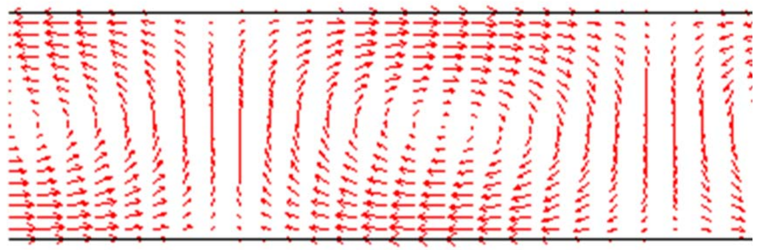
$$\bar{H}(y,z) = \left[ \bar{y}H_1 \sin(k_y y) + \bar{z}H_2 \cos(k_y y) \right] e$$

$$k_y = \frac{m\pi}{d}$$

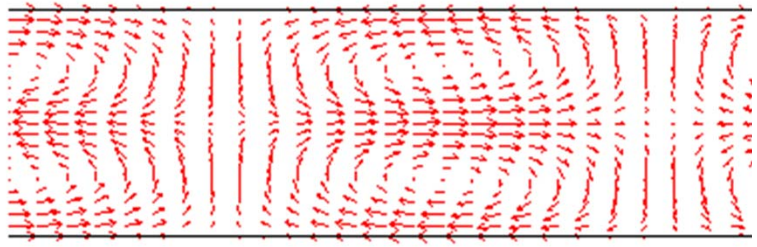
Each *m* gives different solution

➔ mode: $TE_m$

TE$_1$




TE$_2$



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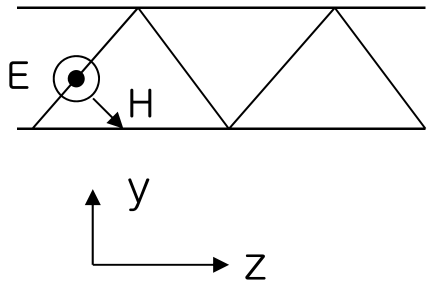


TE Solution

$$\bar{E}(y,z) = \bar{x}E_0 \sin(k_y y)e^{-j\beta z} \qquad k_y = \frac{m\pi}{d}$$

Each mode has its own propagation constant $\beta$

$$\beta = \sqrt{k^2 - k_y{}^2} = \sqrt{\omega^2\mu\varepsilon - \frac{m^2\pi^2}{d^2}}$$

How many modes for a given waveguide at $\omega$?

For given *m*, there is $\omega$ for which $\beta$=0

$$\omega_c = \frac{m\pi}{d\sqrt{\mu\varepsilon}} \quad \text{or} \quad f_c = \frac{m}{2d\sqrt{\mu\varepsilon}},$$

➔ cut-off frequency

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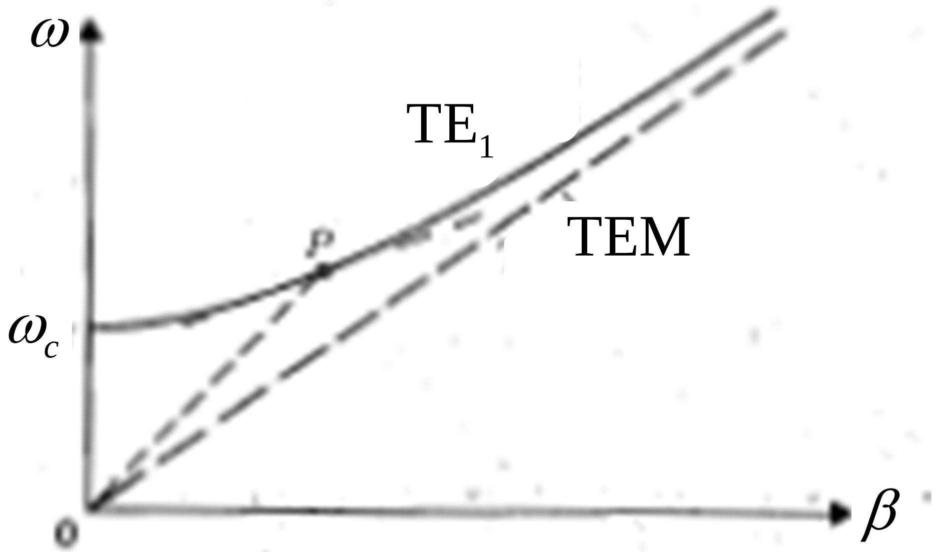
$$\beta = \sqrt{k^2 - k_y^2} = \sqrt{\omega^2 \mu\varepsilon - \frac{m^2\pi^2}{d^2}}$$

$\omega$ vs $\beta$ diagram




- Phase velocity?

- Group velocity?

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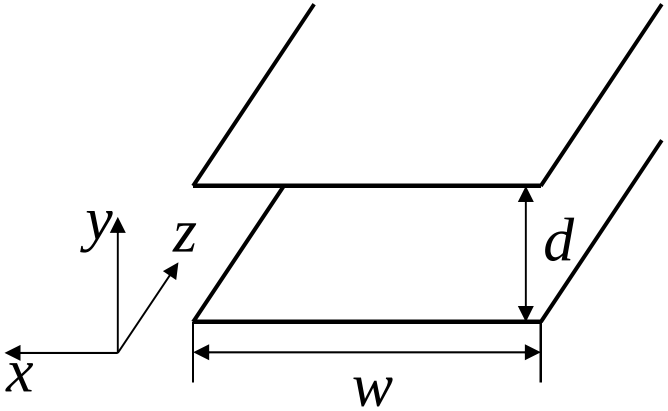


Wave equation in magnetic field?

$$\frac{d^2}{dy^2}\bar{H}(y) + (k^2 - \beta^2)\bar{H}(y) = 0$$

$$\bar{H}(y) = \bar{x}H_0\cos(k_y y) \quad (k^2 = k_y{}^2 + \beta^2)$$

$$\bar{H}(y,z) = \bar{x}H_0\sin(k_y y)e^{-j\beta z}$$

Boundary conditions? $\quad k_y = \frac{m\pi}{d}$ $\quad$ Quantization in $k_y$ and $\beta$

Corresponding electric field?

$$\bar{E}(y,z) = \bar{y}E_1\cos(k_y y)e^{-j\beta z} + \bar{z}E_2\sin(k_y y)e^{-j\beta z}$$

➔ TM Solution

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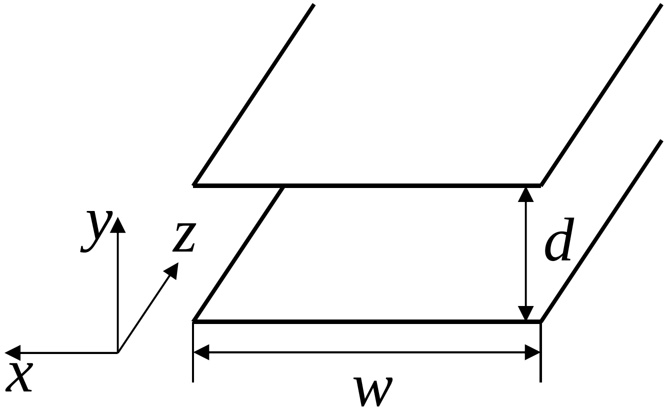
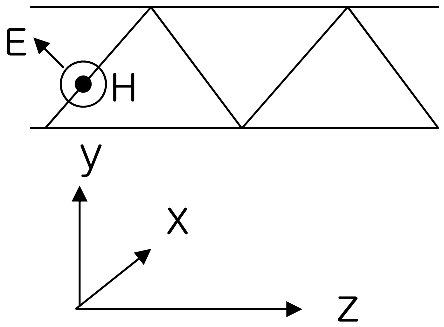



TM Solution

$$\bar{H}(y,z) = \bar{x}H_0 \cos(k_y y)e^{-j\beta z} \qquad k_y = \frac{n\pi}{d}$$

Interpretation $\cos(k_y y)e^{-j\beta z} \sim (e^{jk_y y} + e^{-jk_y y})e^{-j\beta z}$

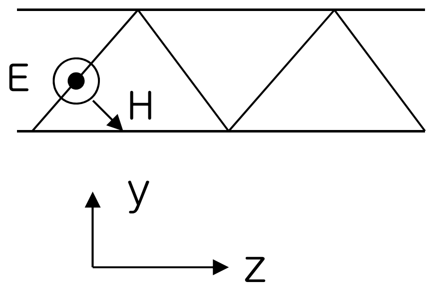
➔ two plane waves propagation in different directions




(parallel polarization)

In y-direction, same phase after one round-trip

$$\exp\left(-j2k_y d\right) = 1 \quad \Rightarrow \quad k_y = \frac{2m\pi}{2d} = \frac{m\pi}{d}$$




TM Solution

$$\bar{H}(y,z)=\bar{x}H_0\cos(k_y y)e^{-j\beta z}$$

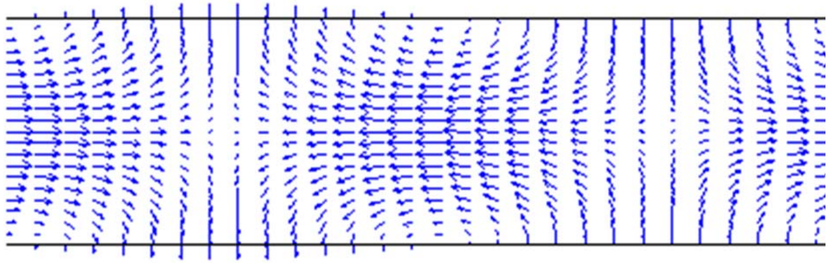
$$\bar{E}(y,z)=\Big[\bar{y}E_1\cos(k_y y)+\bar{z}E_2\sin(k_y y$$

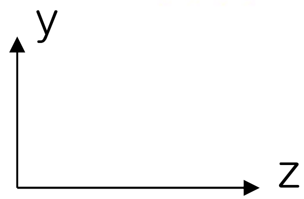
$$k_y=\frac{m\pi}{d}$$

Each $m$ gives different solution

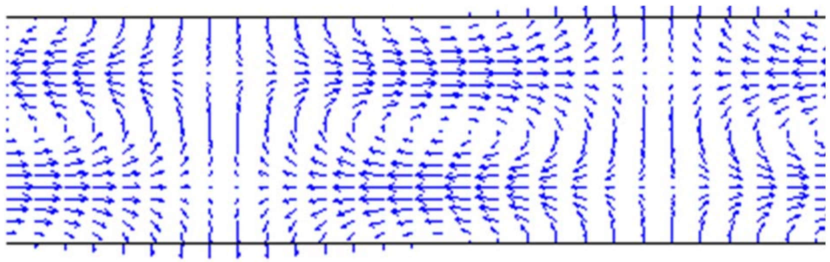
➔ mode: $TM_m$

$TM_1$






$TM_2$



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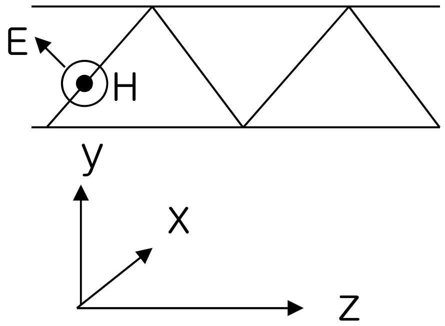


TM Solution

$$\bar{H}(y,z) = \bar{x}H_0 \cos(k_y y)e^{-j\beta z} \qquad k_y = \frac{n\pi}{d}$$

Each mode has its own propagation constant $\beta$

$$\beta = \sqrt{k^2 - k_y^{\ 2}} = \sqrt{\omega^2 \mu\varepsilon - \frac{n^2\pi^2}{d^2}}$$

For a given waveguide, there is a finite number of guided modes at $\omega$

For given *m*, there is $\omega$ for which $\beta$=0

$$\omega_c = \frac{m\pi}{d\sqrt{\mu\varepsilon}} \quad \text{or} \quad f_c = \frac{m}{2d\sqrt{\mu\varepsilon}},$$

➔ cut-off frequency

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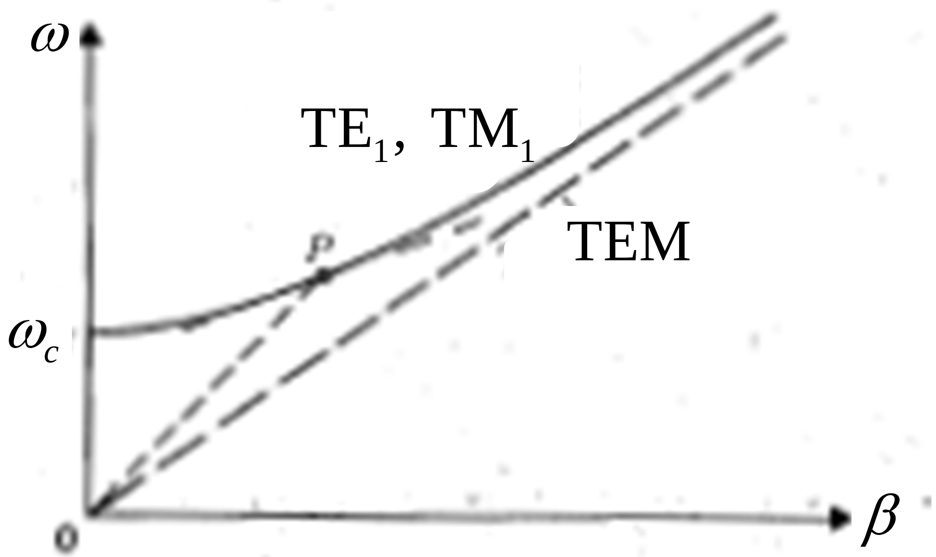
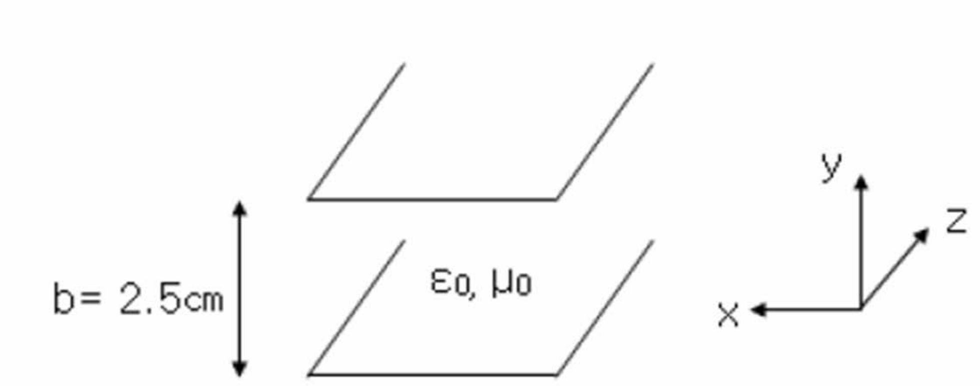
$$\beta = \sqrt{k^2 - k_y{}^2} \;\; = \;\; \sqrt{\omega^2 \mu\varepsilon - \frac{m^2\pi^2}{d^2}}$$

For a given waveguide, $TE_m$ and $TM_m$ have the same $\omega$-$\beta$ relation



## Homework

Consider a lossless parallel-plate waveguide shown below.




**(a)** How many TE or TM modes are there that can propagate in the waveguide if the EM wave frequency is 30 GHz?

**(b)** Find the expression for E(x,y,z) of $TE_2$ mode.