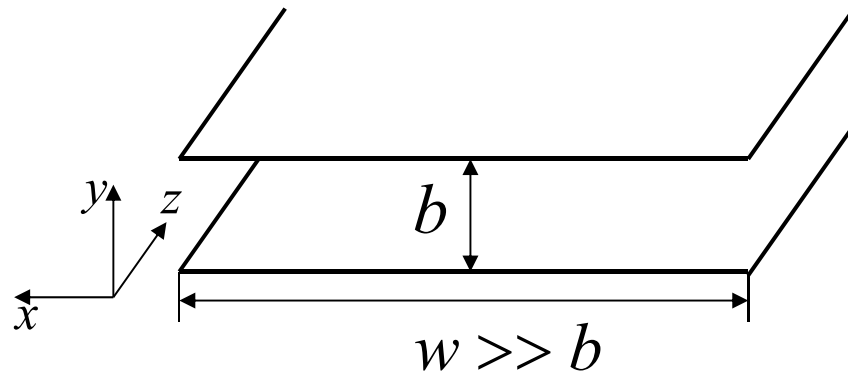
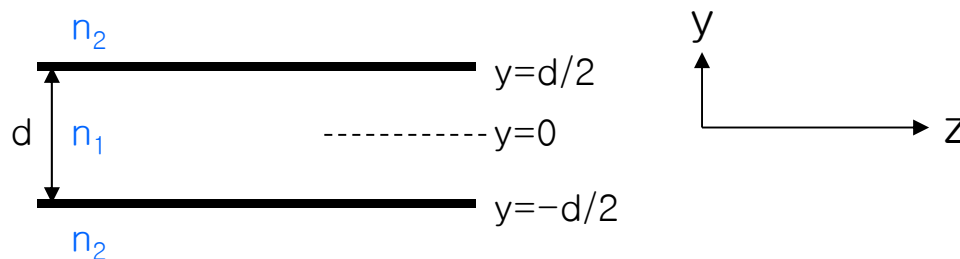


Lect. 14 Rectangular Metallic Waveguide



2-D waveguides

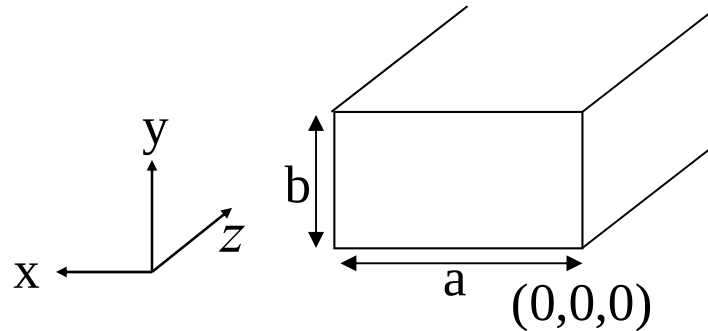
- TE, TM modes
- TE: $E_z = 0$, H_z non-zero
- TM: non-zero E_z , $H_z = 0$
- k_y is quantized



3-D waveguides ?

Lect. 14 Rectangular Metallic Waveguide

Rectangular metallic waveguide



$$\nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2}{\partial t^2} \bar{E} = 0$$

$$\nabla^2 \bar{H} - \mu\epsilon \frac{\partial^2}{\partial t^2} \bar{H} = 0$$

$$\bar{E}(x, y, z, t) = \bar{E}(x, y) \cdot e^{-j\beta z} \cdot e^{j\omega t}$$

$$\bar{H}(x, y, z, t) = \bar{H}(x, y) \cdot e^{-j\beta z} \cdot e^{j\omega t}$$

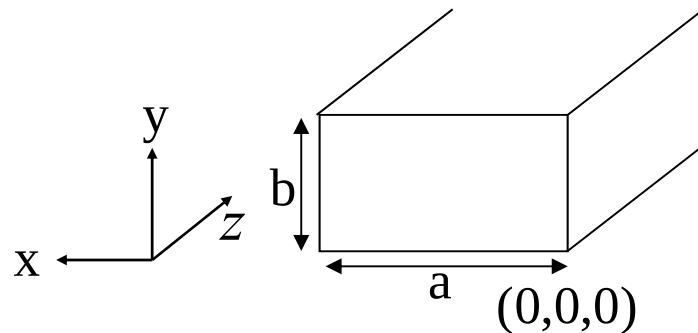
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \bar{E}(x, y) + (k^2 - \beta^2) \bar{E}(x, y) = 0 \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \bar{H}(x, y) + (k^2 - \beta^2) \bar{H}(x, y) = 0$$

Solve for E_z with $H_z=0$
(TM)

Solve for H_z with $E_z=0$
(TE)

Lect. 14 Rectangular Metallic Waveguide

Rectangular Metallic Waveguide: TE modes



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) H_z(x, y) + (k^2 - \beta^2) H_z(x, y) = 0$$

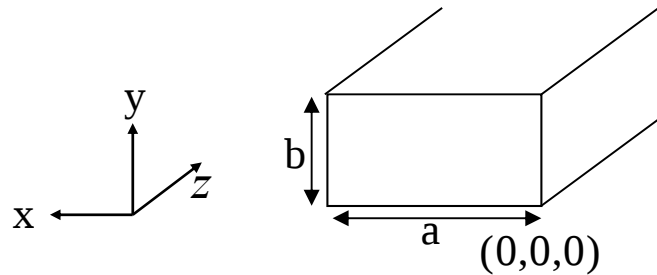
$H_z(x, y) = X(x) \cdot Y(y)$ Separation of variables

$$\frac{\partial^2}{\partial x^2}(XY) + \frac{\partial^2}{\partial y^2}(XY) + (k^2 - \beta^2)(XY) = 0 \quad \frac{1}{X} \frac{\partial^2}{\partial x^2} X + \frac{1}{Y} \frac{\partial^2}{\partial y^2} Y + (k^2 - \beta^2) = 0$$

$$\frac{1}{X} \frac{\partial^2}{\partial x^2} X + \frac{1}{Y} \frac{\partial^2}{\partial y^2} Y + (k_x^2 + k_y^2) = 0 \quad (k^2 = k_x^2 + k_y^2 + \beta^2)$$

$$\frac{d^2}{dx^2} X + k_x^2 X = 0 \quad \frac{d^2}{dy^2} Y + k_y^2 Y = 0$$

Lect. 14 Rectangular Metallic Waveguide



$$H_z(x, y) = X(x)Y(y),$$

$$\frac{d^2}{dx^2} X + k_x^2 X = 0$$

$$\frac{d^2}{dy^2} Y + k_y^2 Y = 0$$

$$X = A \sin(k_x x) + B \cos(k_x x)$$

$$Y = C \sin(k_y y) + D \cos(k_y y)$$

$$\left. \frac{\partial X}{\partial x} \right|_{x=0} = 0 \Rightarrow A = 0$$

$$\left. \frac{\partial Y}{\partial y} \right|_{y=0} = 0 \Rightarrow C = 0$$

$$\left. \frac{\partial X}{\partial x} \right|_{x=a} = 0 \Rightarrow k_x a = m\pi, \quad k_x = \frac{m\pi}{a}$$

$$\left. \frac{\partial Y}{\partial y} \right|_{y=b} = 0 \Rightarrow k_y b = n\pi, \quad k_y = \frac{n\pi}{b}$$

$$X = B \cos(k_x x)$$

$$Y = D \cos(k_y y)$$

$$\therefore H_z(x, y) = H_0 \cos(k_x x) \cos(k_y y) \quad k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad \beta^2 = k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

Lect. 14 Rectangular Metallic Waveguide

From $H_z(x, y) = H_0 \cos(\frac{m\pi}{a}x) \cos(\frac{n\pi}{b}y)$, it can be derived (TE_{mn} mode)

$$E_x(x, y) = \frac{j\omega\mu}{h^2} \left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$$

$$E_y(x, y) = -\frac{j\omega\mu}{h^2} \left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

$$H_x(x, y) = \frac{j\beta}{h^2} \left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

$$H_y(x, y) = \frac{j\beta}{h^2} \left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \quad \left[h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]$$

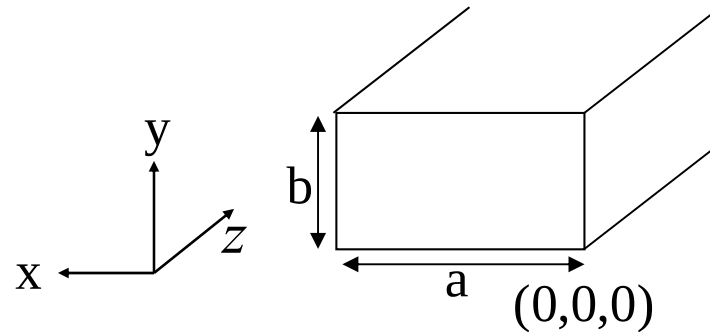
$$\beta = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{\omega^2 \mu \epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad f_c = \frac{1}{2\sqrt{\mu \epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Dominant TE mode?
(Smallest f_c , or largest β)

TE₀₁ or TE₁₀ mode

Lect. 14 Rectangular Metallic Waveguide

TM mode



$$\nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2}{\partial t^2} \bar{E} = 0$$

$$\bar{E}(x, y, z, t) = \bar{E}(x, y) \cdot e^{-j\beta z} \cdot e^{j\omega t}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \bar{E}(x, y) + (k^2 - \beta^2) \bar{E}(x, y) = 0$$

Solve for E_z with $H_z=0$

Lect. 14 Rectangular Metallic Waveguide

$$E_z(x, y) = E_0 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \quad \text{TM}_{mn} \text{ Mode}$$

$$E_x(x, y) = -\frac{j\beta}{h^2} \left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$$

$$E_y(x, y) = -\frac{j\beta}{h^2} \left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

$$H_x(x, y) = \frac{j\omega\epsilon}{h^2} \left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

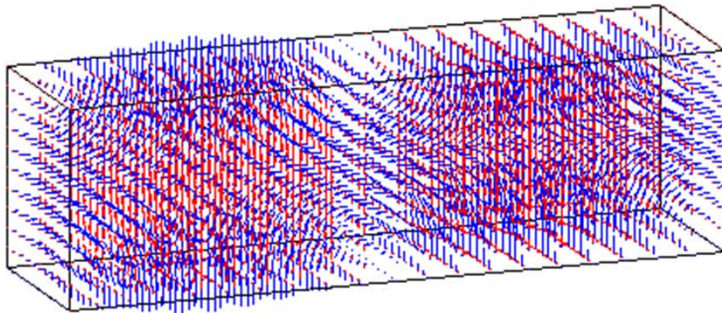
$$H_y(x, y) = -\frac{j\omega\epsilon}{h^2} \left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \quad \left[h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]$$

$$\beta = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{\omega^2 \mu \epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad f_c = \frac{1}{2\sqrt{\mu \epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

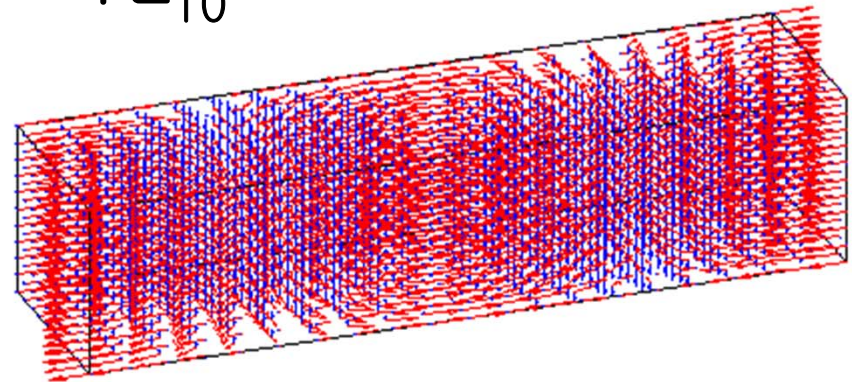
Dominant TM mode? TM_{11} Mode

Lect. 14 Rectangular Metallic Waveguide

TM_{11}

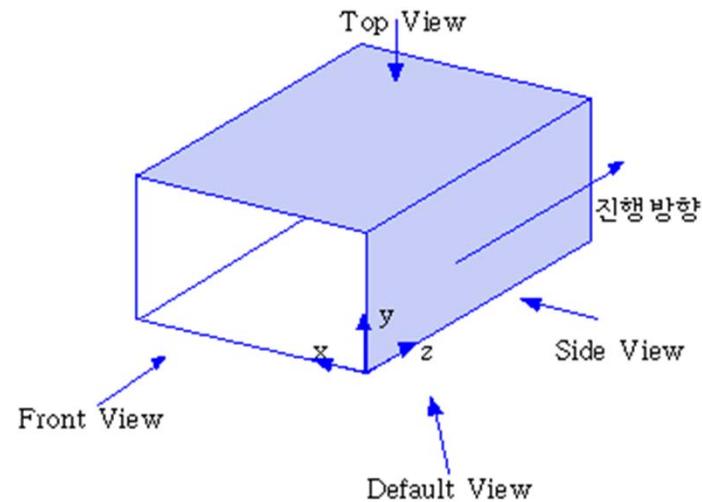


TE_{10}



Lect. 14 Rectangular Metallic Waveguide

Wave propagation for TE_{10} and TM_{11} modes in rectangular waveguide



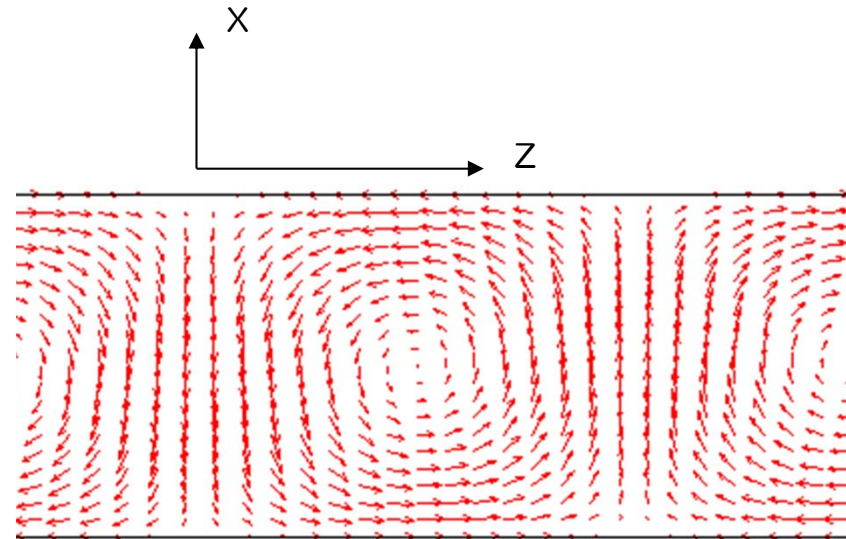
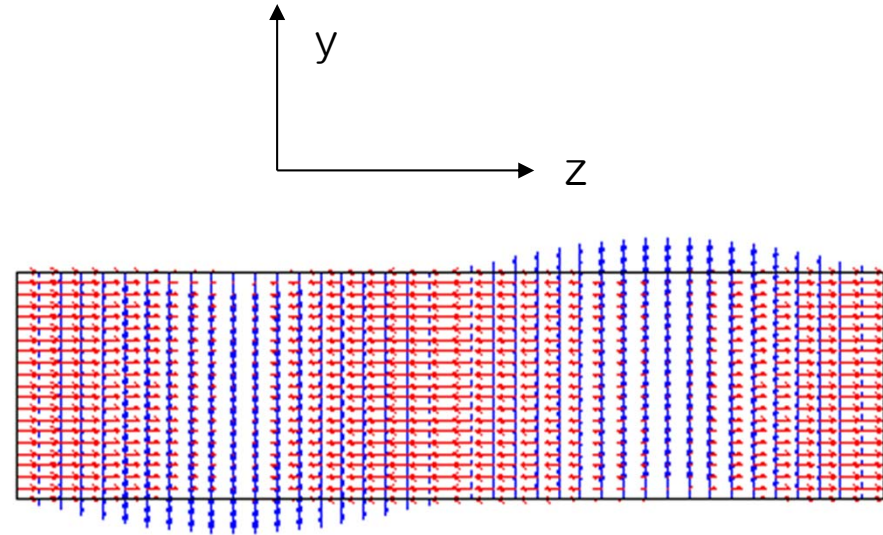
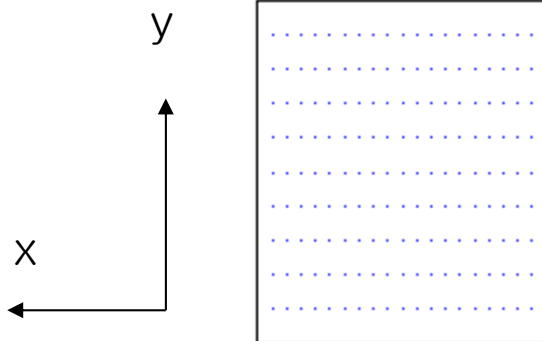
Lect. 14 Rectan

TE₁₀

$$E_y(x, y, t) = \frac{\omega\mu}{h^2} \left(\frac{\pi}{a}\right) H_0 \sin\left(\frac{\pi}{a}x\right) \sin(\omega t - \beta z)$$

$$H_x(x, y, t) = -\frac{\beta}{h^2} \left(\frac{\pi}{a}\right) H_0 \sin\left(\frac{\pi}{a}x\right) \sin(\omega t - \beta z)$$

$$H_z(x, y, t) = H_0 \cos\left(\frac{\pi}{a}x\right) \cos(\omega t - \beta z)$$



Lect. 14 Rectang

TM₁₁

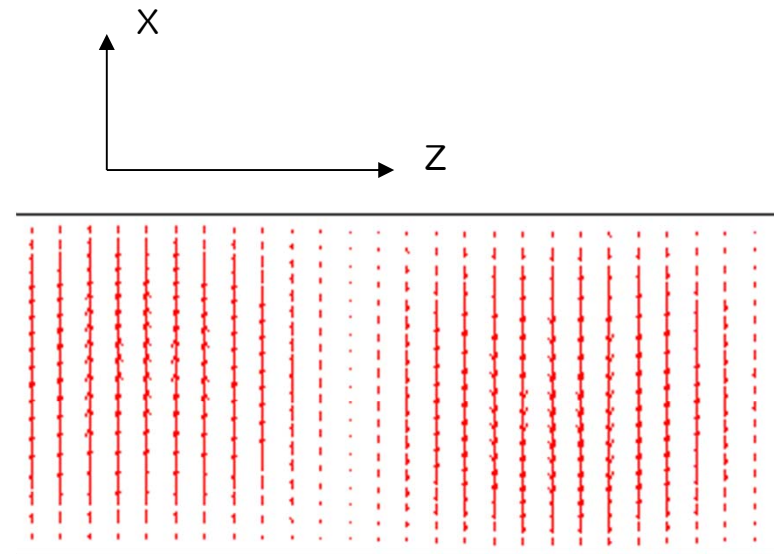
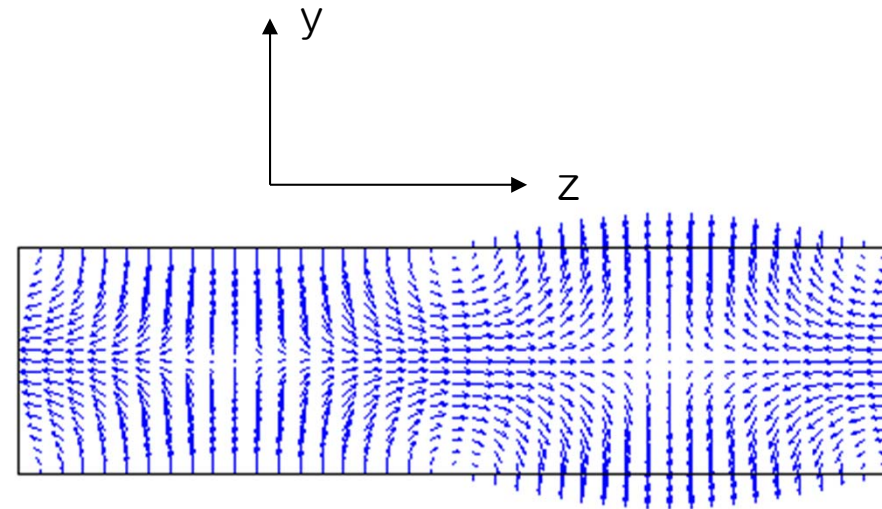
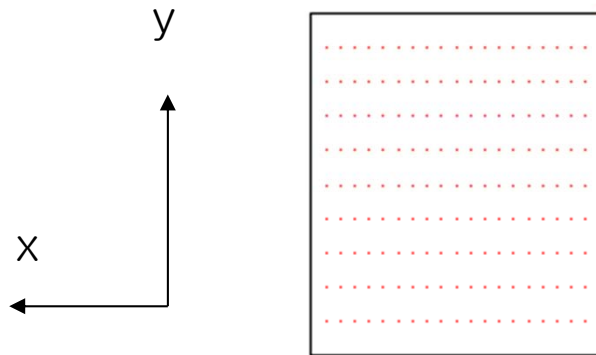
$$E_x(x, y, t) = \frac{\beta}{h^2} \left(\frac{\pi}{a} \right) E_0 \cos\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z)$$

$$E_y(x, y, t) = \frac{\beta}{h^2} \left(\frac{\pi}{b} \right) E_0 \sin\left(\frac{\pi}{a} x\right) \cos\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z)$$

$$E_z(x, y, t) = E_0 \sin\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{b} y\right) \cos(\omega t - \beta z)$$

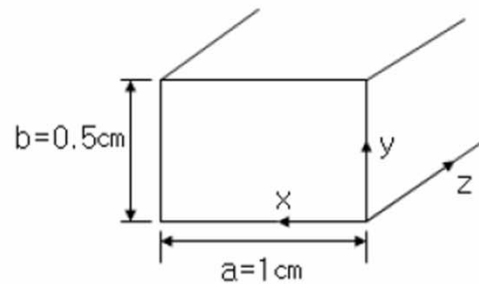
$$H_x(x, y, t) = -\frac{\omega \epsilon}{h^2} \left(\frac{\pi}{b} \right) E_0 \sin\left(\frac{\pi}{a} x\right) \cos\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z)$$

$$H_y(x, y, t) = \frac{\omega \epsilon}{h^2} \left(\frac{\pi}{a} \right) E_0 \cos\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z)$$



Lect. 14 Rectangular Metallic Waveguide

Consider an empty rectangular waveguide with dimensions 1 cm x 0.5 cm shown below. The waves are propagating in z-direction.



(a,b) Identify the waveguide modes that have the field profiles shown below. In the figure, electric fields are represented by solid lines and magnetic fields by dashed lines.

