Uncertainty principle and particle current

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Outline

1 Uncertainty principle

2 Particle current

3 Summary

Uncertainty principle

Uncertainty principle

❖ Uncertainty

- By using measurable property → position, momentum

❖ Momentum operator

$$\hat{p} = -ih\nabla$$

with $\nabla \equiv \mathbf{x}_o \frac{\partial}{\partial x} + \mathbf{y}_o \frac{\partial}{\partial y} + \mathbf{z}_o \frac{\partial}{\partial z}$

x_0, y_0, z_0 unit vector of each direction

Uncertainty principle

❖ Momentum operator

$$\frac{\hat{p}^2}{2m} \equiv -\frac{\hbar^2}{2m}\nabla^2$$

- Classical notation of energy

$$E = \frac{p^2}{2m} + V$$

- Hamiltonian operator of Schrödinger equation

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V = \frac{\hat{p}^2}{2m} + V$$

Uncertainty principle

❖ **Example of position-momentum uncertainty**

- Gaussian wavepacket

$$\Psi_G(z,t) \propto \int_k \exp\left[-\left(\frac{k-\bar{k}}{2\Delta k}\right)^2\right]\exp\left\{-i\left[\omega(k)t - kz\right]\right\}dk$$

at time t = 0 as

$$\Psi(z,0) = \int_k \Psi_k(k)\exp(ikz)\,dk$$

where

$$\Psi_k(k) \propto \exp\left[-\left(\frac{k-\bar{k}}{2\Delta k}\right)^2\right]$$

$\Psi_k(k)$ is the representation of the wavefunction in k space

Uncertainty principle

❖ Example of position-momentum uncertainty

- With wavefunction in K space

$$\Psi_k(k) \propto \exp\left[-\left(\frac{k-\bar{k}}{2\Delta k}\right)^2\right]$$

- Probability of finding value $\hbar k$ for the momentum

$$P_k = \left|\Psi_k(k)\right|^2 \propto \exp\left[-\frac{\left(k-\bar{k}\right)^2}{2\left(\Delta k\right)^2}\right]$$

corresponds to statistical Gaussian probability distribution with standard deviation Δk

Uncertainty principle

❖ Example of position-momentum uncertainty

- Gaussian wavepacket at time t=0

$$\Psi(z,0)=\int_k \Psi_k(k)\exp(ikz)\,dk$$

→ Fourier transform of $\Psi_k(k)$

$$\Psi(z,0)\propto \exp\left[-(\Delta k)^2 z^2\right]$$

Modulus square

$$\left|\Psi(z,0)\right|^2 \propto \exp\left[-2(\Delta k)^2 z^2\right]$$

Uncertainty principle

❖ **Example of position-momentum uncertainty**

- Standard deviation in momentum space

$$\left|\Psi(z,0)\right|^2 \propto \exp\left[-2(\Delta k)^2 z^2\right]$$

$$\|$$

- Standard deviation in space

$$\left|\Psi(z,0)\right|^2 \propto \exp\left[-\frac{z^2}{2(\Delta z)^2}\right]$$

$$\downarrow$$

$$\Delta k \Delta z = 1/2$$

Uncertainty principle

❖ Example of position-momentum uncertainty

- Multiplying $\hbar$ to each term

$$\Delta p \Delta z = \frac{\hbar}{2}$$

→ Gaussian pulse will broaden in space as it propagates, even though the range of k values remains the same

- Gaussian shape is the minimum possible product of these standard deviations

$$\Delta p \Delta z \geq \frac{\hbar}{2}$$

→ Uncertainty principle

Uncertainty principle

❖ **Uncertainty principle to fourier analysis**

$$\Delta\omega\Delta t \geq \frac{1}{2}$$

→ One cannot simultaneously have both a well defined frequency and a well defined time for a signal

Particle current

Particle current

❖ **Divergence of vector F**

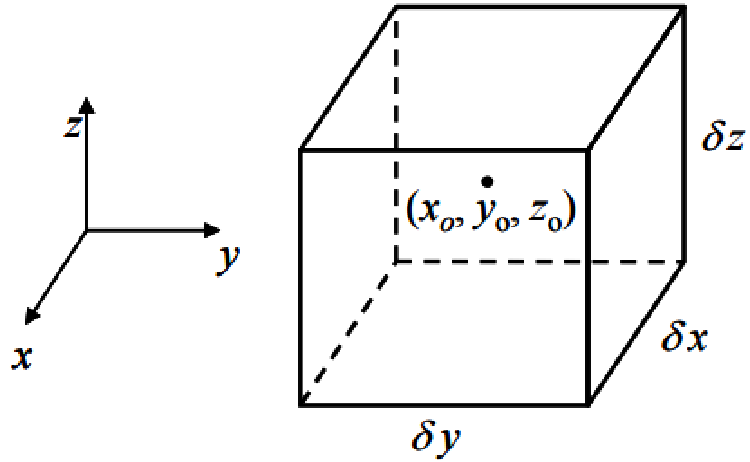
$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

❖ **Visualize F**

- Flow of some quantity, like mass or charge, through a small cuboidal little box
- Flow leaves the box at front

$$F_x(x_o + \delta x/2, y_o, z_o)\delta y \delta z$$

$$F_x\left(x_o + \frac{\delta x}{2}, y_o, z_o\right)\delta y \delta z \equiv \mathbf{F}\left(x_o + \frac{\delta x}{2}, y_o, z_o\right)\cdot \delta \mathbf{A}_{yz}$$



- Flow arrives the box at back

$$F_x(x_o - \delta x/2, y_o, z_o)\delta y \delta z$$

Particle current

❖ **Net amount leaving the box through the front or back**

$$F_x\left(x_o+\frac{\delta x}{2},y_o,z_o\right)\delta y\delta z-F_x\left(x_o-\frac{\delta x}{2},y_o,z_o\right)\delta y\delta z$$

$$=\frac{F_x\left(x_o+\frac{\delta x}{2},y_o,z_o\right)-F_x\left(x_o-\frac{\delta x}{2},y_o,z_o\right)}{\delta x}\delta x\delta y\delta z$$

$$\simeq\frac{\partial F_x}{\partial x}\delta x\delta y\delta z$$

❖ **Total amount of flow leaving small box dividing by *δ*V**

$$\operatorname{div}\mathbf{F}=\nabla\cdot\mathbf{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$$

Particle current

❖ **Classically, conservation of flow of particles**

$$\frac{\partial s}{\partial t} = -\nabla . \mathbf{j}_p$$

s particle density
j_p particle current density

❖ **s = |Ψ(r,t)|²**

$$\frac{\partial \Psi(\mathbf{r},t)}{\partial t} = \frac{1}{i\hbar} \hat{H} \Psi(\mathbf{r},t) \qquad \frac{\partial \Psi^*(\mathbf{r},t)}{\partial t} = -\frac{1}{i\hbar} \hat{H}^* \Psi^*(\mathbf{r},t)$$

$$\frac{\partial}{\partial t}\left[\Psi^* \Psi\right] = \Psi^* \frac{\partial \Psi}{\partial t} + \Psi \frac{\partial \Psi^*}{\partial t}$$

$$\frac{\partial}{\partial t}\left[\Psi^* \Psi\right] + \frac{i}{\hbar}\left(\Psi^* \hat{H} \Psi - \Psi \hat{H}^* \Psi^*\right) = 0$$

Particle current

❖ **Potential *V* is real and does not depend in time, Hamiltonian to be the form**

$$\hat{H} \equiv -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

$$\Psi^*\hat{H}\Psi - \Psi\hat{H}^*\Psi^* = -\frac{\hbar^2}{2m}\left[\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right] + \Psi^* V\Psi - \Psi V\Psi^*$$

$$= -\frac{\hbar^2}{2m}\left[\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right]$$

$$\frac{\partial}{\partial t}\left[\Psi^*\Psi\right] + \frac{i}{\hbar}\left(\Psi^*\hat{H}\Psi - \Psi\hat{H}^*\Psi^*\right) = 0$$

$$\frac{\partial}{\partial t}\left[\Psi^*\Psi\right] - \frac{i\hbar}{2m}\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) = 0$$

Particle current

❖ **Using algebraic trick**

$$\Psi\nabla^2\Psi^* - \Psi^*\nabla^2\Psi = \Psi\nabla^2\Psi^* + \nabla\Psi\nabla\Psi^* - \nabla\Psi\nabla\Psi^* - \Psi^*\nabla^2\Psi$$
$$= \nabla\cdot\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right)$$

$$\frac{\partial}{\partial t}\left[\Psi^*\Psi\right] - \frac{i\hbar}{2m}\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) = 0$$

$$\frac{\partial\left(\Psi^*\Psi\right)}{\partial t} = -\frac{i\hbar}{2m}\nabla\cdot\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right)$$

modulus square of wave function

Particle current

❖ **Particle current**

$$j_p = \frac{i\hbar}{2m}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)$$

❖ **For nth energy eigenstate**

$$\Psi_n(\mathbf{r},t) = \exp\left(-i\frac{E_n}{\hbar}t\right)\psi_n(\mathbf{r})$$

- **Particle current**

$$\mathbf{j}_{pn}(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\Psi_n(\mathbf{r},t)\nabla\Psi_n^*(\mathbf{r},t) - \Psi_n^*(\mathbf{r},t)\nabla\Psi_n(\mathbf{r},t)\right)$$

Particle current

- **Gradient has no effect on time factor → particle current**

$$\mathbf{j}_{pn}(\mathbf{r},t)=\frac{i\hbar}{2m}\exp\left(-i\frac{E_n}{\hbar}t\right)\exp\left(i\frac{E_n}{\hbar}t\right)\left(\psi_n(\mathbf{r})\nabla\psi_n^*(\mathbf{r})-\psi_n^*(\mathbf{r})\nabla\psi_n(\mathbf{r})\right)$$
$$=\frac{i\hbar}{2m}\left(\psi_n(\mathbf{r})\nabla\psi_n^*(\mathbf{r})-\psi_n^*(\mathbf{r})\nabla\psi_n(\mathbf{r})\right)$$

❖ **Does not depends on time**

$$\mathbf{j}_{pn}(\mathbf{r},t)=\mathbf{j}_{pn}(\mathbf{r})$$

- Particle current is constant in any energy eigenstate
- For real spatial eigenfunctions particle current is actually zero

Summary

❖ Uncertainty principle

$$\Delta p \Delta z \geq \frac{\hbar}{2}$$

- One cannot simultaneously have both a well defined real space and a well defined momentum space for a signal

❖ Particle current

- Particle current is constant in any energy eigenstate
- Electron in eigenstate does not radiate electromagnetic radiation

Thank you
for your attention !!