Functions and Dirac notation

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Functions as vectors

- **Functions**

$$x_1 \rightarrow f(x_1)$$

$$x_2 \rightarrow f(x_2)$$

- **Vector of function**

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \end{bmatrix}$$

- We do this for many values of x to represent another calculation result.

Functions as vectors

- **Probability density function**

$$\int |f(x)|^2 \, dx \cong \begin{bmatrix} f^*(x_1) & f^*(x_2) & f^*(x_3) & \cdots \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \end{bmatrix} \delta x$$

- The integration can be approximated to the product of row vector and column vector.
- Row vector and column vector can be express by simple notation.

Dirac “bra-ket” notation

- **Dirac “bra-ket” represents a row vector and a column vector**

- **Notation with column vector of a function**

- “ket” notation $\longrightarrow$ $$|f(x)\rangle \equiv \begin{bmatrix} f(x_1)\sqrt{\delta x} \\ f(x_2)\sqrt{\delta x} \\ f(x_3)\sqrt{\delta x} \\ \vdots \end{bmatrix}$$

where $\delta x \to 0$, $\sqrt{\delta x}$ for normalization

< “ket” notation refers to a column vector of a function. >

Dirac "bra-ket" notation

- **Similarly, "bra" notation is defined.**

  - "bra" notation is a version of row vector from a function.

$$\langle f(x)| \equiv \left[f^*(x_1)\sqrt{\delta x} \quad f^*(x_2)\sqrt{\delta x} \quad f^*(x_3)\sqrt{\delta x} \quad \cdots \right]$$

where $\delta x \to 0$, $\sqrt{\delta x}$ for normalization

- Note that "bra" notation is a vector with complex conjugated components.
- Both "bra-ket" notations refer to the exactly same function.

Dirac "bra-ket" notation

- **Assume a row vector**

$$\begin{bmatrix} a_1^* & a_2^* & a_3^* & \cdots \end{bmatrix}$$

Then, we call this row vector as the
- Hermitian adjoint
- Hermitian transpose
- adjoint

$\longrightarrow$ of a column vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix}$

Dirac "bra-ket" notation

- **This Hermitian adjoint is expressed by the character "†" as a superscription.**

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix}^\dagger = \begin{bmatrix} a_1^* & a_2^* & a_3^* & \cdots \end{bmatrix}$$

- **From this relation, the "bra" notation is the Hermitian adjoint of the "ket" notation.**

$$(|f\rangle)^\dagger = \langle f| \qquad (\langle f|)^\dagger = |f\rangle$$

Dirac "bra-ket" notation

- **If we consider $f(x)$ as a vector and add "bra-ket" notation,**

$$\int \left|f(x)\right|^2 dx \equiv \left[f^*(x_1)\sqrt{\delta x} \quad f^*(x_2)\sqrt{\delta x} \quad f^*(x_3)\sqrt{\delta x} \quad \cdots \right] \begin{bmatrix} f(x_1)\sqrt{\delta x} \\ f(x_2)\sqrt{\delta x} \\ f(x_3)\sqrt{\delta x} \\ \vdots \end{bmatrix}$$

$$\equiv \sum_n f^*(x_n)\sqrt{\delta x}\, f(x_n)\sqrt{\delta x}$$

$$\equiv \langle f(x) | f(x) \rangle$$

So, the integration can be expressed as the "bra-ket" notation

Dirac "bra-ket" notation

- **Expansion to a quantum mechanical wave function,**

$$f(x) = \sum_n c_n \psi_n(x)$$

we can write the function as the "bra"

$$\langle f(x)| \equiv \begin{bmatrix} c_1^* & c_2^* & c_3^* & \cdots \end{bmatrix}$$

and the "ket"

$$|f(x)\rangle \equiv \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix}$$

Dirac "bra-ket" notation

- **We can evaluate**

$$\int \left|f(x)\right|^2 dx \equiv \int f^*(x) f(x) dx \equiv \int \left[\sum_n c_n^* \psi_n^*(x)\right]\left[\sum_m c_m \psi_m(x)\right] dx$$

$$\equiv \sum_{n,m} c_n^* c_m \int \psi_n^*(x) \psi_m(x) dx \equiv \sum_{n,m} c_n^* c_m \delta_{nm} \equiv \sum_n \left|c_n\right|^2$$

$$\equiv \begin{bmatrix} c_1^* & c_2^* & c_3^* & \cdots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix} \equiv \langle f(x) | f(x) \rangle$$

So, we can express the calculation more simply.

Q & A