Functions and Dirac notation

Park Chun Taek

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School of Electrical and Electronic Engineering, Yonsei University, Korea





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Functions as vectors

Functions

$$x_1 \to f(x_1)$$
$$x_2 \to f(x_2)$$

Vector of function

$$f(x_1)$$

$$f(x_2)$$

$$f(x_3)$$

$$\vdots$$

- We do this for many values of x to represent another calculation result.



Functions as vectors

• Probability density function

$$\int |f(x)|^2 dx \cong \left[f^*(x_1) \quad f^*(x_2) \quad f^*(x_3) \quad \cdots \right] \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \end{bmatrix} \delta x$$

- The integration can be approximated to the product of row vector and column vector.
- Row vector and column vector can be express by simple notation.



- Dirac "bra-ket" represents a row vector and a column vector
- Notation with column vector of a function

- "ket" notation
$$\longrightarrow$$
 $|f(x)\rangle \equiv \begin{bmatrix} f(x_1)\sqrt{\delta x} \\ f(x_2)\sqrt{\delta x} \\ f(x_3)\sqrt{\delta x} \\ \vdots \end{bmatrix}$

where $\delta x \to 0$, $\sqrt{\delta x}$ for normalization

< "ket" notation refers to a column vector of a function. >



- Similarly, "bra" notation is defined.
 - "bra" notation is a version of row vector from a function.

$$\langle f(x)| \equiv [f^*(x_1)\sqrt{\delta x} \quad f^*(x_2)\sqrt{\delta x} \quad f^*(x_3)\sqrt{\delta x} \quad \cdots]$$

where $\delta x \to 0$, $\sqrt{\delta x}$ for normalization

- Note that "bra" notation is a vector with complex conjugated components.
- Both "bra-ket" notations refer to the exactly same function.



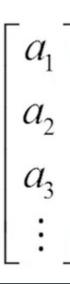
Assume a row vector

$$\begin{bmatrix} a_1^* & a_2^* & a_3^* & \cdots \end{bmatrix}$$

Then, we call this row vector as the

- Hermitian adjoint
- Hermitian transpose
- adjoint

→ of a column vector





• This Hermitian adjoint is expressed by the character "†" as a superscription.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix}^{\dagger} = \begin{bmatrix} a_1^* & a_2^* & a_3^* & \cdots \end{bmatrix}$$

• From this relation, the "bra" notation is the Hermitian adjoint of the "ket" notation.

$$(|f\rangle)^{\dagger} = \langle f| \qquad (\langle f|)^{\dagger} = |f\rangle$$



• If we consider f(x) as a vector and add "bra-ket" notation,

$$\int |f(x)|^2 dx = \left[f^*(x_1) \sqrt{\delta x} \quad f^*(x_2) \sqrt{\delta x} \quad f^*(x_3) \sqrt{\delta x} \quad \dots \right] \begin{bmatrix} f(x_1) \sqrt{\delta x} \\ f(x_2) \sqrt{\delta x} \\ f(x_3) \sqrt{\delta x} \end{bmatrix}$$

$$\equiv \sum_n f^*(x_n) \sqrt{\delta x} f(x_n) \sqrt{\delta x}$$

$$\equiv \langle f(x) | f(x) \rangle$$

So, the integration can be expressed as the "bra-ket" notation



• Expansion to a quantum mechanical wave function,

$$f(x) = \sum_{n} c_n \psi_n(x)$$

we can write the function as the "bra"

$$\langle f(x)| \equiv \begin{bmatrix} c_1^* & c_2^* & c_3^* & \cdots \end{bmatrix}$$

and the "ket"

$$\left| f(x) \right\rangle \equiv \begin{vmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{vmatrix}$$



We can evaluate

$$\int |f(x)|^2 dx = \int f^*(x) f(x) dx = \int \left[\sum_n c_n^* \psi_n^*(x) \right] \left[\sum_m c_m \psi_m(x) \right] dx$$

$$= \sum_{n,m} c_n^* c_m \int \psi_n^*(x) \psi_m(x) dx = \sum_{n,m} c_n^* c_m \delta_{nm} = \sum_n |c_n|^2$$

$$= \left[c_1^* \quad c_2^* \quad c_3^* \quad \cdots \right] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix} = \langle f(x) | f(x) \rangle$$

So, we can express the calculation more simply.



Q & A

