The time-dependent Schrödinger equation

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Rationalizing the time-dependent equation

- The time-dependent Schrödinger equation for a particle with mass m is

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r},t)\Psi(\mathbf{r},t)=i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$$

with $E=\hbar\omega$, $k=\sqrt{2mE/\hbar^2}$ and $V=0$

- Then the solution is

$$\exp\left[-i(\omega t\pm kz)\right]\equiv\exp\left[-i\left(\frac{Et}{\hbar}\pm kz\right)\right]\equiv\exp\left(-i\frac{Et}{\hbar}\right)\exp(\mp ikz)$$

Positive direction

- The wave is

$$\exp\left[i(kz-Et/\hbar)\right]$$

Rationalizing the time-dependent equation

The time-independent equation

- Extraction of time-dependence using separation of variable

$$\Psi(\mathbf{r},t)=\psi(\mathbf{r})\exp(-iEt/\hbar)$$

- Substitute the new sol into Schrödinger equation, then

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r})\Psi(\mathbf{r},t)$$
$$=-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r})\exp(-iEt/\hbar)+V(\mathbf{r})\psi(\mathbf{r})\exp(-iEt/\hbar)$$
$$=\left[-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r})+V(\mathbf{r})\psi(\mathbf{r})\right]\exp(-iEt/\hbar)=E\psi(\mathbf{r})\exp(-iEt/\hbar)$$
$$=E\Psi(\mathbf{r},t)$$

Free parameter

- In doing this, we can get the wave equation via particular energy E with appropriate $V(r)$

Rationalizing the time-dependent equation

Oscillations and time-independence

$$\left|\Psi(\mathbf{r},t)\right|^2 = \left[\exp(+iEt/\hbar)\psi^*(\mathbf{r})\right]\times\left[\exp(-iEt/\hbar)\psi(\mathbf{r})\right] = \left|\psi(\mathbf{r})\right|^2$$

- Wave density of the wave function is stable in time

$$\int \left|\Psi(\mathbf{r},t)\right|^2 dr = 1$$

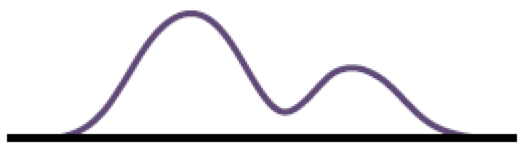
- by the normalization condition

$$\Psi(\mathbf{r},t) = 0 \quad \text{as} \quad \mathrm{r} \to \infty$$

- Since Integral must be finite, solution have to satisfies normalization condition

Solution of the time-dependent equation

Contrast to classical wave equation



$f(z-ct)$?

$g(z+ct)$?

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r},t)\Psi(\mathbf{r},t)=i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$$

- In Schrödinger's equation, for a known potential *V*, if we knew the wave function at every point in space at some time t_0
- we could evaluate the left hand side of the equation at that time for all **r**
- so we would know $\partial\Psi(\mathbf{r},t)/\partial t$ for all **r**
- so we could integrate the equation to deduce $\Psi(\mathbf{r},t)$ at all future times

properties

Linear superposition

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r},t)\Psi(\mathbf{r},t)=i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$$

- The time-dependent Schrödinger equation is linear

$$\Psi_c(\mathbf{r},t)=c_a\Psi_a(\mathbf{r},t)+c_b\Psi_b(\mathbf{r},t)$$

- The fact that $\Psi_c(\mathbf{r},t)$ is a solution if $\Psi_a(\mathbf{r},t)$ and $\Psi_b(\mathbf{r},t)$ are solutions is the property of linear superposition where c_a and c_b complex constant

properties

expansion in Eigen states

$$\Psi(\mathbf{r},0)=\psi(\mathbf{r})=\sum a_n\psi_n(\mathbf{r})$$

- By the time-independent equation & superposition

$$\Psi_n(\mathbf{r},t)=\exp(-iE_nt/\hbar)\psi_n(\mathbf{r})$$

- By separation of time-dependence

$$\Psi(\mathbf{r},t)=\sum_n a_n\Psi_n(\mathbf{r},t)=\sum_n a_n\exp(-iE_nt/\hbar)\psi_n(\mathbf{r})$$

- Hence, if we expand the wave function at time t=0 In the energy eigenstates
- we have solved for the time evolution of the state just by adding up the above sum