Si Photonics

Lecture 2: Electro-Magnetic Waves

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Lecture 2: Electro-Magnetic Waves

Maxwell's Equations

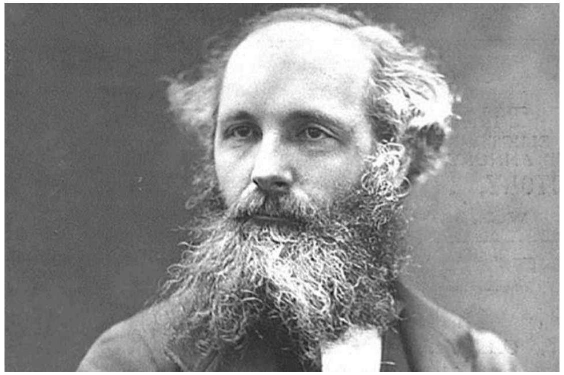
$$\nabla \cdot \overline{D} = \rho$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \overline{D} = \varepsilon \overline{E}$$

$$\nabla \cdot \overline{B} = 0 \qquad \overline{B} = \mu \overline{H}$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$$

(Published in 1861,1862)



James Clerk Maxwell
(1831-1879)

$\overline{E}$: Electric Field (V/m)

$\overline{D}$: Electric Displacement, Electric Flux (C/m^2)

$\overline{H}$: Magnetic Field (A/m)

$\overline{B}$: Magnetic Flux (T = V-s/m2)

ρ : Charge Density (C/m^3)

$\overline{J}$: Current Density (A/m^2)

ε : Permittivity (F/m)

μ : Permeability (H/m)

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$$\nabla \cdot \overline{D} = \rho$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

$$\nabla \cdot \overline{B} = 0$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$$

$$\overline{D} = \varepsilon \overline{E}$$

$$\overline{B} = \mu \overline{H}$$

➔ Equations governing electro-magnetics

$$\overline{E}, \overline{D}, \overline{H}, \overline{B}, \overline{J} : \ \overline{x} f_x(x,y,z,t) + \overline{y} f_y(x,y,z,t) + \overline{z} f_z(x,y,z,t)$$

$$\rho, \varepsilon, \mu : \ f(x,y,z)$$

✓ Simplification

1) source free medium ➔ $\rho = 0, \ \overline{J} = 0$

2) uniform medium ➔ $\varepsilon, \mu \neq f(x,y,z)$

$$\nabla \times \left(\nabla \times \overline{E}\right) = -\nabla \times \left(\frac{\partial \overline{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \overline{B}) = -\mu \frac{\partial}{\partial t}(\nabla \times \overline{H})$$

$$= -\mu \frac{\partial^2 \overline{D}}{\partial t^2} = -\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}$$

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$$\nabla\times\left(\nabla\times\overline{E}\right)=-\mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2}\qquad \overline{E}=\overline{x}E_x(x,y,z)+\overline{y}E_y(x,y,z)+\overline{z}E_z(x,y,z)$$

$$\nabla\times\left(\nabla\times\overline{E}\right)=\nabla\cancel{\left(\nabla\cdot\overline{E}\right)}-\nabla^2\overline{E}$$

$$\nabla^2\overline{E}=\ \overline{x}\nabla^2E_x+\overline{y}\nabla^2E_y+\overline{z}\nabla^2E_z$$

(Laplacian)

$$\nabla^2 f(x,y,z)=\frac{\partial^2 f(x,y,z)}{\partial x^2}+\frac{\partial^2 f(x,y,z)}{\partial y^2}+\frac{\partial^2 f(x,y,z)}{\partial z^2}$$

$$\text{From }\nabla\cdot\overline{D}=\rho,\qquad \nabla\cdot\varepsilon\overline{E}=\rho\qquad \varepsilon\nabla\cdot\overline{E}=\rho\qquad \nabla\cdot\overline{E}=0$$

$$\boxed{\therefore\nabla^2\overline{E}=\mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2}}\qquad\left(\nabla^2\overline{H}=\mu\varepsilon\frac{\partial^2\overline{H}}{\partial t^2}\right)$$

➔ EM Wave Equation

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$$\nabla^2 \overline{E}(x,y,z,t) = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}(x,y,z,t)$$

An example of wave equation solutions: $\overline{E} = \overline{x} E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon}\, z)\right]$

Verification: $\nabla^2 \overline{E} \implies \nabla^2 E_x = \dfrac{d^2}{dz^2} E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon}\, z)\right]$

$$= -\omega^2 \mu\varepsilon E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon}\, z)\right]$$

$$\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \implies \mu\varepsilon \frac{\partial^2}{\partial t^2} E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon}\, z)\right]$$

$$= -\omega^2 \mu\varepsilon E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon}\, z)\right]$$

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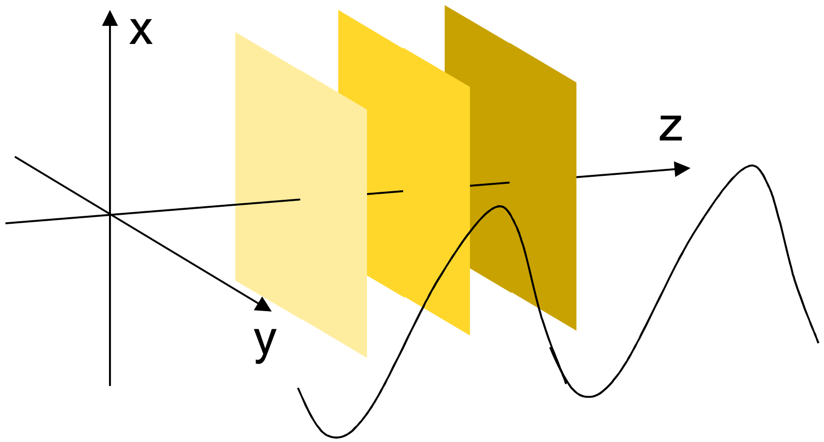
$$\overline{E} = \overline{x} E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon} z) \right]$$

How does this look?

At a given t,



E-field on any x-y plane is uniform

➔ Plane wave

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$\overline{E} = \overline{x}E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon}z)\right]$ How does this look?

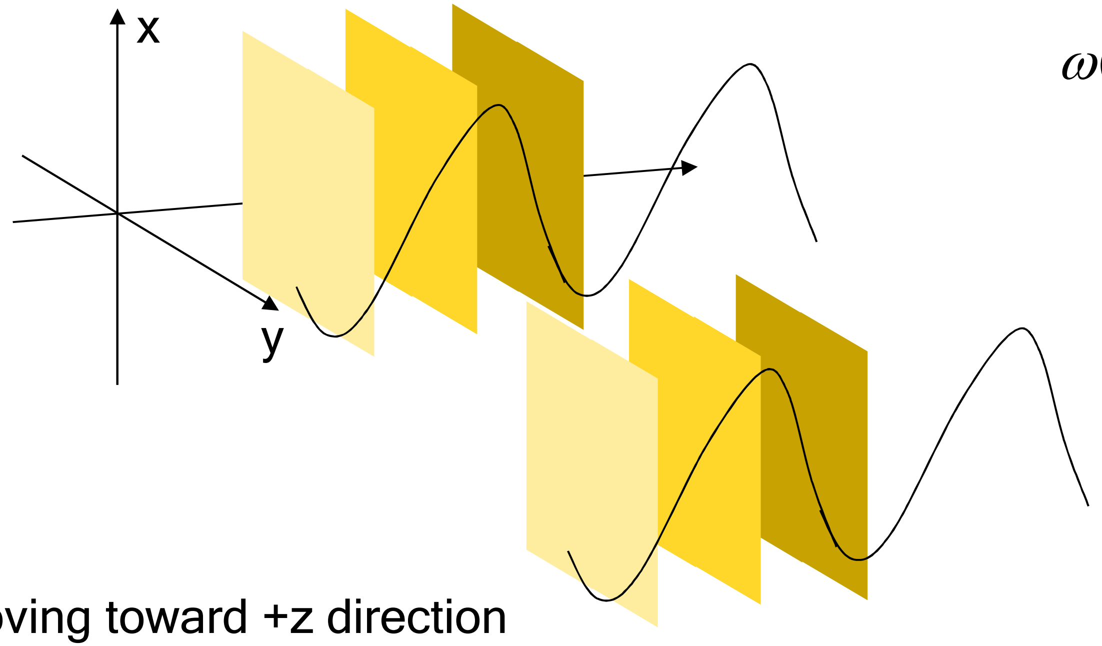
At two different t's

For the constant phase

$$\Delta\left[\omega(t - \sqrt{\mu\varepsilon}z)\right] = 0$$

$$\omega(\Delta t - \sqrt{\mu\varepsilon}\Delta z) = 0$$

$$\therefore \frac{\Delta z}{\Delta t} = \frac{1}{\sqrt{\mu\varepsilon}}$$



Phase velocity of EM wave

Moving toward +z direction

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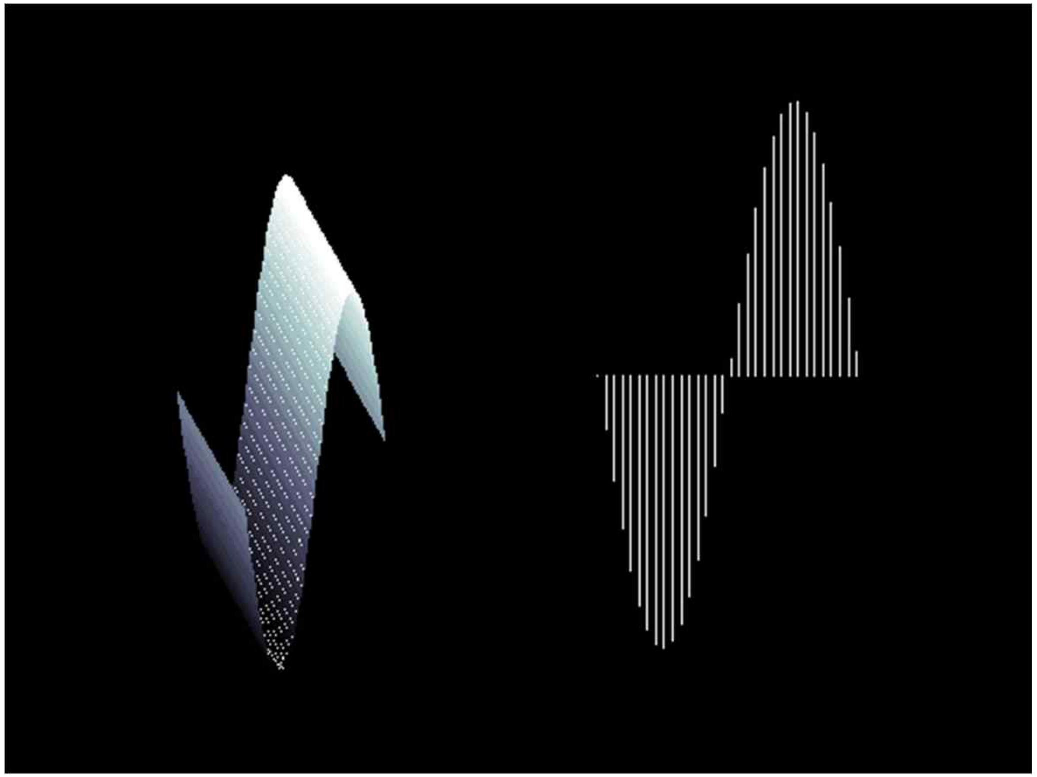
$$\overline{E} = \overline{x}E_0 \cos\left[\omega(t - \sqrt{\mu\varepsilon}z)\right]$$



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Solve the wave equation mathematically $\nabla^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}$

Assume time dependence of $e^{j\omega t}$ (Time harmonic solutions)

$\overline{E}(x, y, z, t) = \overline{E}(x, y, z) \cdot e^{j\omega t}$ (Separation of variables)

$$\nabla^2 \overline{E}(x, y, z) \cdot e^{j\omega t} + \mu\varepsilon\omega^2 \overline{E}(x, y, z) \cdot e^{j\omega t} = 0$$

$$\nabla^2 \overline{E}(x, y, z) + \mu\varepsilon\omega^2 \overline{E}(x, y, z) = 0$$

Let $\omega\sqrt{\mu\varepsilon} = k$

$\nabla^2 \overline{E}(x, y, z) + k^2 \overline{E}(x, y, z) = 0$ Time-Independent EM Wave equation

$k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$ k : wave number $\left[\mathrm{m}^{-1}\right]$ $\left(\omega = 2\pi f = \frac{2\pi}{T}\right)$

spacial frequency

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$$\nabla^2 \overline{E}(x,y,z) + k^2 \overline{E}(x,y,z) = 0$$

Assume $\overline{E}(x,y,z) = \overline{x}E(z)$ for simplicity

$$\boxed{\frac{d^2}{dz^2}E(z) + k^2 E(z) = 0}$$

Solutions?

$$E(z) = E_0^+ e^{-jkz} + E_0^- e^{jkz}$$

$$E(z,t) = (E_0^+ e^{-jkz} + E_0^- e^{jkz})e^{j\omega t} = E_0^+ e^{j(\omega t - kz)} + E_0^- e^{j(\omega t + kz)}$$

How does this look? ➔ Take the real part for visualization

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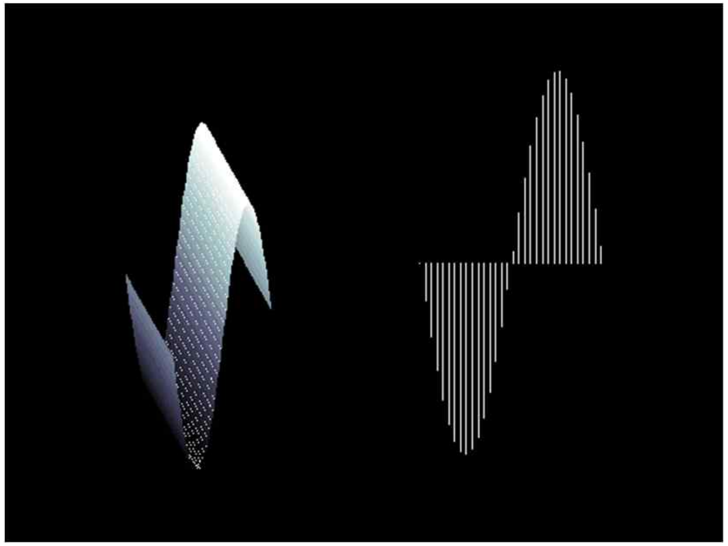
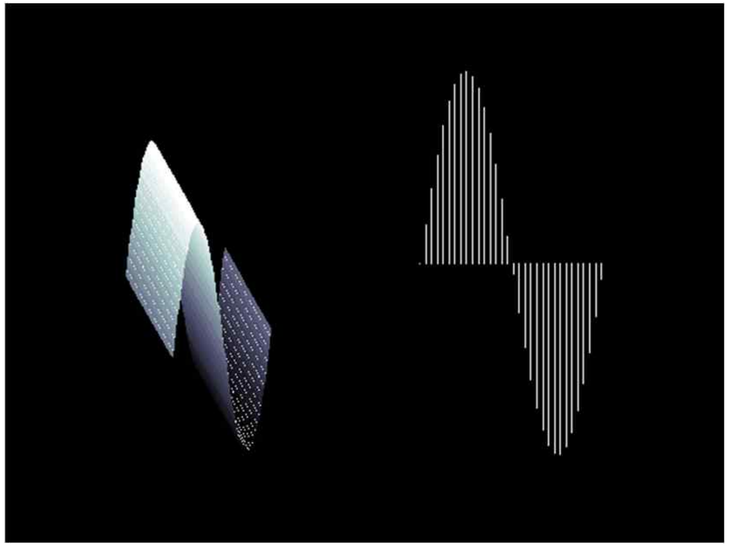
$$\mathrm{Re}\left\{ E_0^+ e^{j(\omega t - kz)} + E_0^- e^{j(\omega t + kz)} \right\} = \ E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz)$$

How does cos(ωt-kz) change as t increases? ➔ Propagation in + z direcition

How does cos(ωt+kz) change as t increases? ➔ Propagation in - zdirecition





Phase velocity? $\Delta\left[\omega t \pm kz)\right] = 0$ $\quad \dfrac{\Delta z}{\Delta t} = \pm\dfrac{\omega}{k} = \pm\dfrac{1}{\sqrt{\mu\varepsilon}}$

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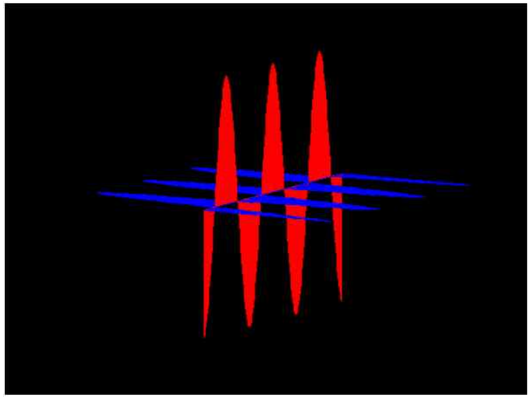
$\overline{E} = \overline{x} E_o e^{-jkz} e^{j\omega t}$ ➔ Corresponding H-field?

From $\nabla \times E = -\dfrac{\partial B}{\partial t}$, $\nabla \times E = -\mu \dfrac{\partial \overline{H}}{\partial t}$ $\overline{H} = -\int \dfrac{\nabla \times \overline{E}}{\mu} dt$

$$\nabla \times E = \begin{vmatrix} \overline{x} & \overline{y} & \overline{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0 e^{-jkz} e^{j\omega t} & 0 & 0 \end{vmatrix} = \overline{y}(-jk) E_0 e^{-jkz} e^{j\omega t}$$



$$\overline{H} = \overline{y} \frac{k}{\omega\mu} E_0 e^{-jkz} e^{j\omega t} = \overline{y} \sqrt{\frac{\varepsilon}{\mu}} E_0 e^{-jkz} e^{j\omega t}$$

H-field has the same wave characteristics (frequency, wave number, propagation direction) but different vector direction and magnitude

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$$\overline{E} = \overline{x} E_o e^{-jkz} e^{j\omega t} \qquad \overline{H} = \overline{y} \sqrt{\frac{\varepsilon}{\mu}} E_0 e^{-jkz} e^{j\omega t}$$

$$\frac{\left|\overline{E}\right|}{\left|\overline{H}\right|} = \sqrt{\frac{\mu}{\varepsilon}} \equiv \eta$$ (Impedance, about 120π or 337 Ω in vacuum)

For plane EM waves,
E-field direction, H-field direction, and propagation direction are orthogonal

$$\overline{H} = \frac{1}{\eta}\left(\overrightarrow{a_k} \times \overrightarrow{E}\right) \qquad \overline{E} = -\eta\left(\overrightarrow{a_k} \times \overrightarrow{H}\right)$$

$$\overline{P} = \overline{E} \times \overline{H}$$ Poynting vector: Propagation of power density

For time-harmonic EM waves, exp(jωt) is often omitted for simplicity

$$\overline{E} = \overline{x} E_o e^{-jkz} \qquad \overline{H} = \overline{y} H_o e^{-jkz}$$

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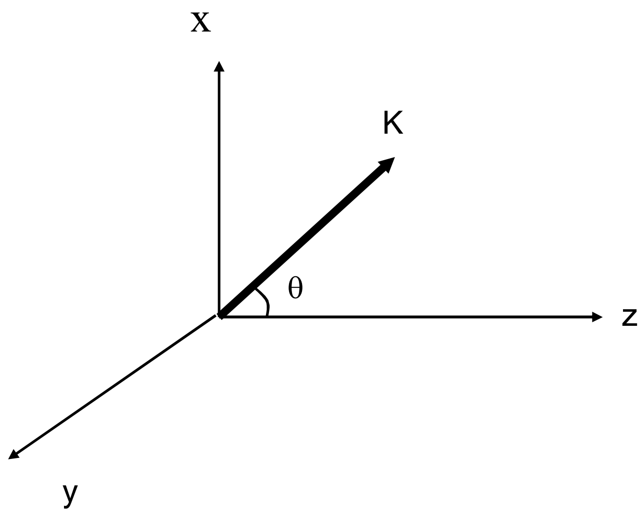
Wave propagating in any direction: Use vector k



$$e^{-jk_x x} \cdot e^{-jk_y y} \cdot e^{-jk_z z} = e^{-j(k_x x + k_y y + k_z z)}$$

$$= e^{-j\bar{k}\cdot\bar{R}}$$

$$\bar{k} = \bar{x}k_x + \bar{y}k_y + \bar{z}k_z$$

$$\bar{R} = \bar{x}x + \bar{y}y + \bar{z}z$$

$\bar{k}$: wave vector

$\angle\bar{k}$: Direction of propagation

$\left|\bar{k}\right|$: $\dfrac{2\pi}{\lambda}$

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Homework
(Due at Sept. 6 11am. Hand in your homework in PDF through LearnUs.
No late hand-ins accepted)

For the plane wave whose E-field is given as $\overline{E} = \overline{y}E_o e^{j(x-z)} e^{j\omega t}$

is propagating in a medium having $\varepsilon=4\varepsilon_0$ and $\mu=\mu_0$.

(a) Which direction does this wave propagate? Give your answer with a unit vector.

(b) What is the frequency in Hz of this wave?

(c) What is the expression for the corresponding H-field?